An Approximate Version of Carathéodory's Theorem and Its Algorithmic Applications

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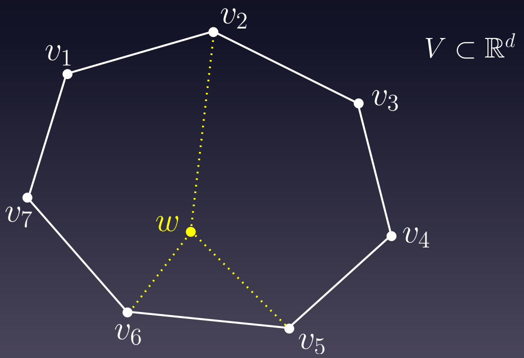
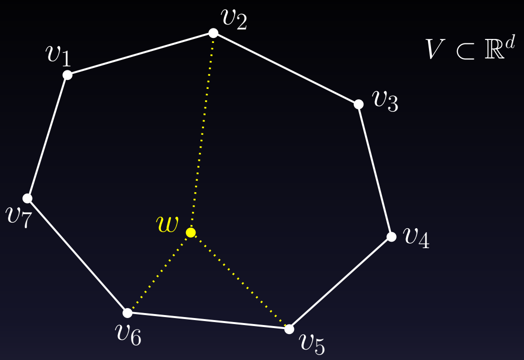
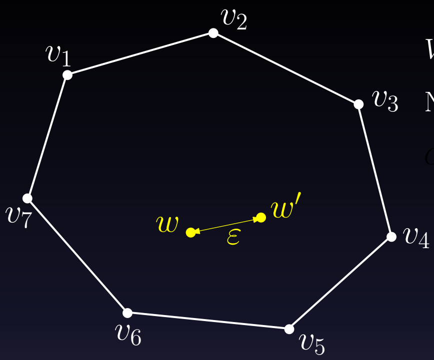
California Institute of Technology

Carathéodory's Theorem

Any vector in the convex hull of a set V in $\mathbb{R}^d$ can be expressed as a convex combination of at most $d+1$ vectors of V.




v_1
v_2
v_3
v_4
v_5
v_6
v_7
w
$V \subset \mathbb{R}^d$



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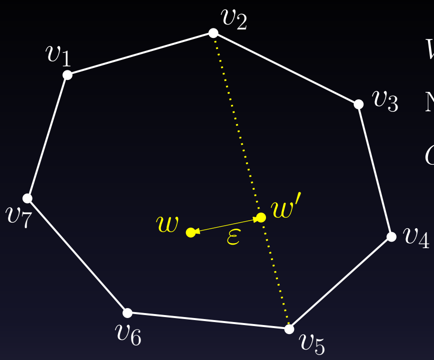
Norm $p \geq 2$

$O\left(\frac{p}{\varepsilon^2}\right)$ vectors



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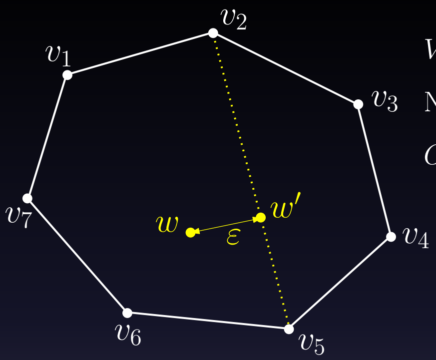

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Approx. Carathéodory's Theorem

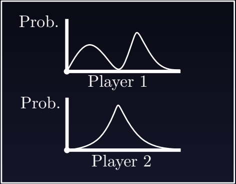
Given set V in the p-unit ball with norm $p \geq 2$, for every vector in the convex hull of V there exists an ε-close (under p-norm distance) vector that is a convex combination of $O\left(\frac{p}{\varepsilon^2}\right)$ vectors of V.

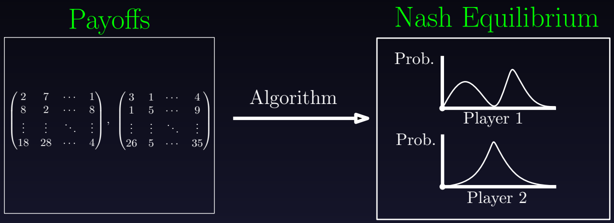
Application: Algorithm for Approximate Nash Equilibria

Payoffs

$$\begin{pmatrix} 2 & 7 & \cdots & 1 \\ 8 & 2 & \cdots & 8 \\ \vdots & \vdots & \ddots & \vdots \\ 18 & 28 & \cdots & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 & \cdots & 4 \\ 1 & 5 & \cdots & 9 \\ \vdots & \vdots & \ddots & \vdots \\ 26 & 5 & \cdots & 35 \end{pmatrix} \xrightarrow{\text{Algorithm}}$$

Nash Equilibrium





Nash equilibrium in two-player games is PPAD-hard [GP06, DGP06, CD06, CDT09].

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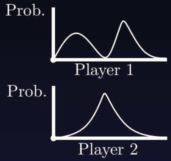

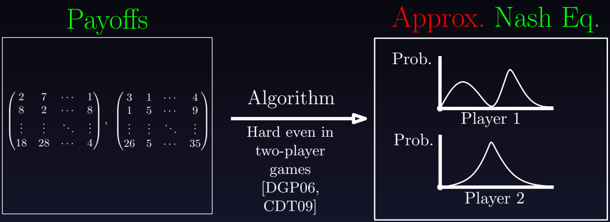
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Prob.
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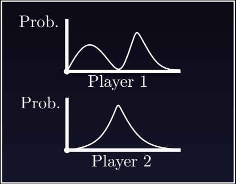

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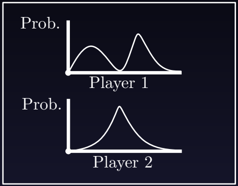

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Payoff matrices A and B of size $n \times n$

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Approximate Nash equilibrium (x, y): No player can benefit more than ε by unilateral deviation

$$\begin{aligned} e_i^T A y &\leq x^T A y + \varepsilon \qquad \forall i \in [n] \ \text{ and} \\ x^T B e_j &\leq x^T B y + \varepsilon \qquad \forall j \in [n] \end{aligned}$$

Computation of Eq. in Two-Player Games

Nash Equilibria

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This Talk: Sparsity

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The sparsity of a game (A, B) is defined to be the maximum number of non-zero entries in any column of $A + B$.

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Theorem

In a two-player s-sparse game an ε-Nash equilibrium can be computed in time $n^{O\left(\log s/\varepsilon^2\right)}$.

Payoff matrices normalized $A, B \in [-1, 1]^{n\times n}$.

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Implications:

- When s is a fixed constant we get a polynomial-time algorithm
- For general games $(s \leq n)$ the running time matches the best-known upper bound: $n^{O(\log n/\varepsilon^2)}$ [LMM'03].

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Note: Sparse Games are Hard [CDT 2006].

Nash eq: $e_i^T A y \le x^T A y \quad \forall i$ and

$$x^T B e_j \le x^T B y \qquad \forall j$$

Bilinear Program for Nash Eq. [MS'64]

$$\begin{array}{rl} \text{maximize} & x^T(A+B)y - \pi_1 - \pi_2 \\ \text{subject to} & x^T B \le \pi_2 \quad \text{and} \quad Ay \le \pi_1 \\ & x, y \in \Delta^n \quad \text{and} \quad \pi_1, \pi_2 \in [-1, 1] \end{array}$$

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Say (x^*, y^*) is a Nash eq. Given $u^* = Cy^*$ we get an LP.

$$\begin{array}{ll} \text{maximize} & x^T u^* - \pi_1 - \pi_2 \\ \text{subject to} & x^T B \le \pi_2 \quad \text{and} \quad Ay \le \pi_1 \\ & x, y \in \Delta^n \quad \text{and} \quad \pi_1, \pi_2 \in [-1, 1] \\ & Cy = u^* \end{array}$$

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A vector *close* to Cy^* is sufficient to find an approx. Nash eq.

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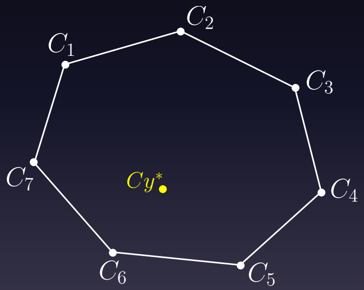

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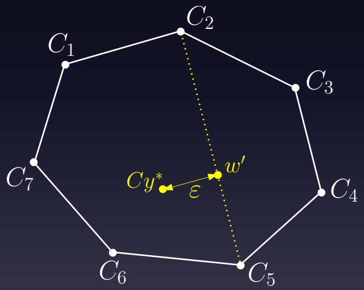

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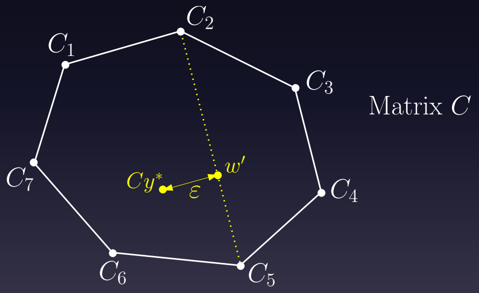

Idea: Exhaustively search for w'

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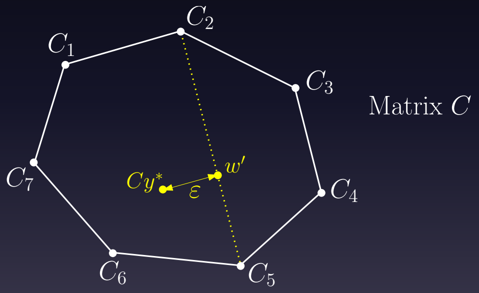

Idea: Exhaustively search for w',
by enumerating subsets of columns of C.

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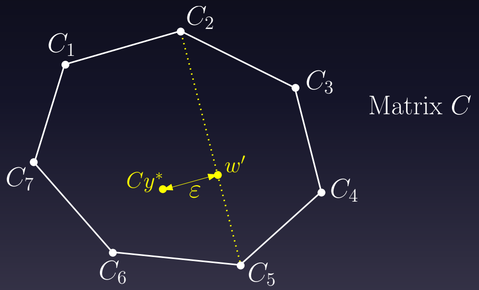

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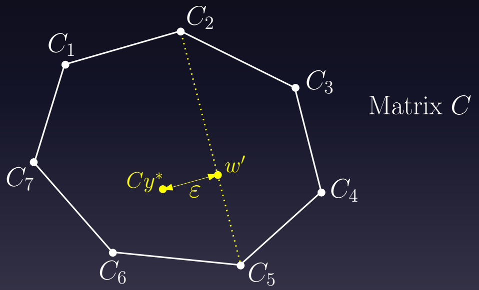
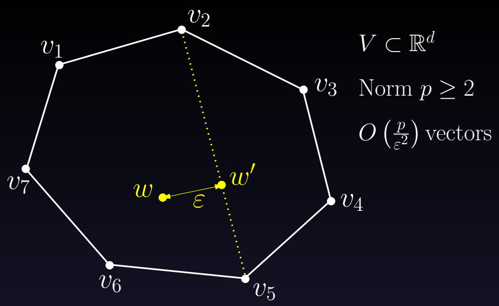


General Result

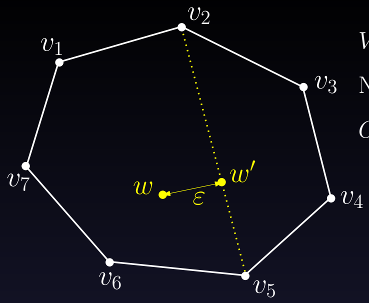
We can efficiently approximate any sparse bilinear or quadratic form over the simplex.

✓ Application: Algorithm for Approximate Nash Equilibria



Approx. Carathéodory's Theorem

Given set V in the p-unit ball with norm $p \geq 2$, for every vector in the convex hull of V there exists an ε-close (under p-norm distance) vector that is a convex combination of $O\left(\frac{p}{\varepsilon^2}\right)$ vectors of V.



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Proof via the Probabilistic Method and Khintchine-Kahane Inequality

Extensions

- Convex hull of matrices with entrywise norm and Schatten p-norm

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Approx. Birkhoff-von Neumann Theorem

For every $d \times d$ doubly stochastic matrix D there exists an ε-close (under the entrywise ∞-norm) doubly stochastic matrix D' such that D' is a convex combination of $O\left(\frac{\log d}{\varepsilon^2}\right)$ permutation matrices.

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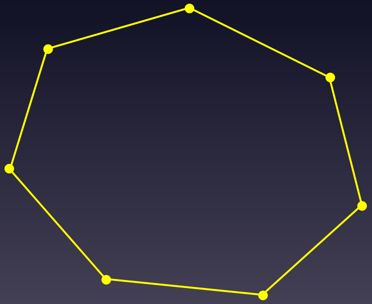

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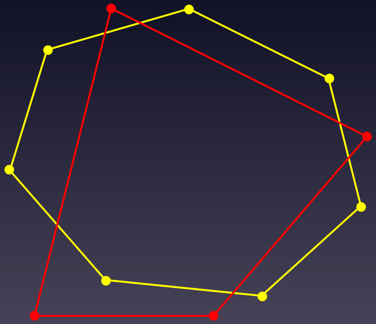

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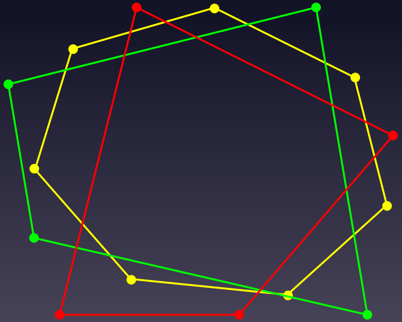

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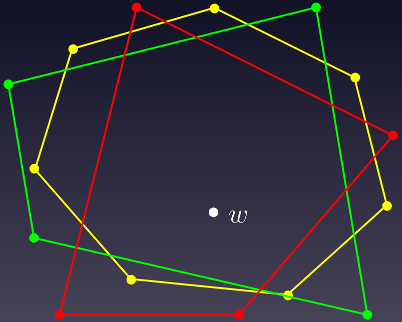

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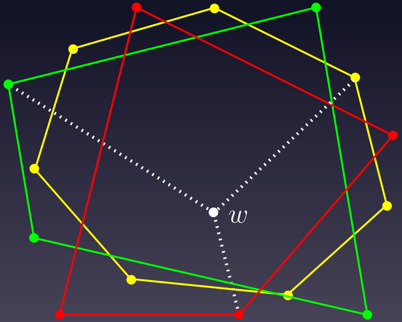

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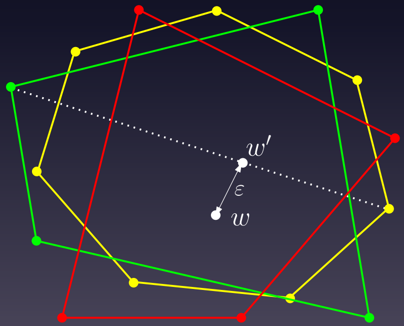

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Barman, An Approximate Version of Carathéodory's Theorem with Applications to Approximating Nash Equilibria and Dense Bipartite Subgraphs, http://arxiv.org/abs/1406.2296, 2014

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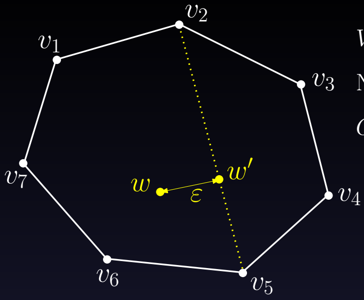

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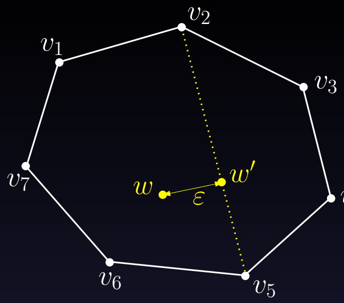

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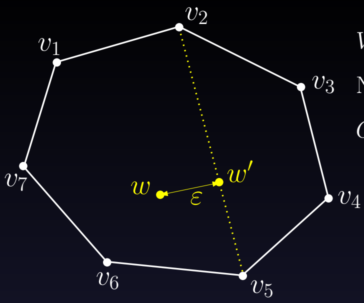
Proof via the Probabilistic Method

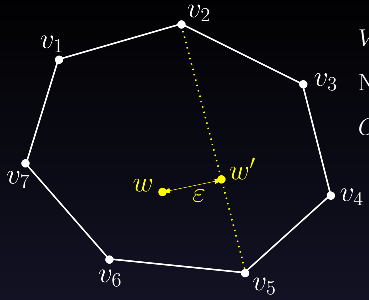


$V \subset \mathbb{R}^d$

Norm $p \geq 2$

$O\left(\frac{p}{\varepsilon^2}\right)$ vectors

$w = \sum_i \alpha_i v_i$ where $\sum_i \alpha_i = 1$ and $\alpha_i \geq 0$



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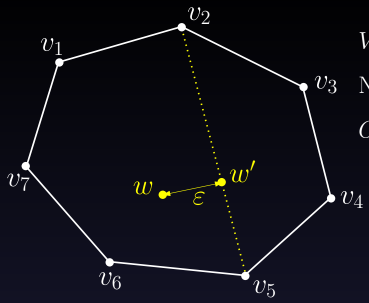
Sample v_i with probability α_i



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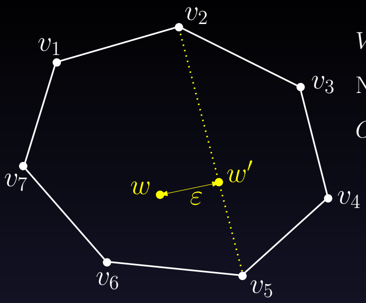
Empirical mean of m i.i.d. samples $s_1, s_2, \ldots, s_m$

$$w = \mathbb{E}\left[\frac{1}{m}\sum_{i=1}^{m} s_i\right]$$



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Consider

$$g(s_1, s_2, \ldots, s_m) := \left\| \frac{1}{m} \sum_{i=1}^{m} s_i - w \right\|_p$$

With i.i.d. samples $s_1, \ldots, s_m \sim w$,
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  (proof via Mcdiarmid's inequality)

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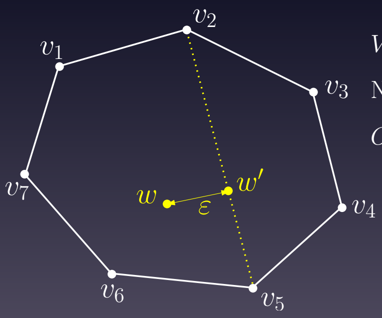

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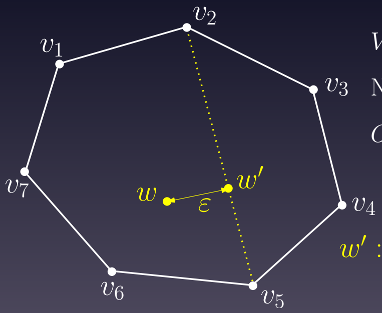

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Norm $p \ge 2$

$O\left(\frac{p}{\varepsilon^2}\right)$ vectors

$w' := \frac{1}{m} \sum_{i=1}^{m} s_i$

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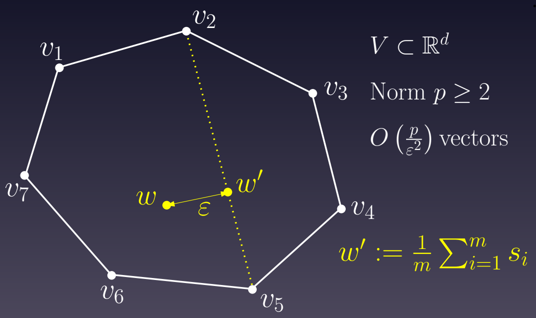
$g(s_1, \ldots, s_m) := \left\| \frac{1}{m} \sum_{i=1}^{m} s_i - w \right\|_p$

- g concentrates: $\Pr\left(|g - \mathbb{E}[g]| > \varepsilon\right) \leq e^{-\varepsilon^2 m}$
- Bounded expectation: $\mathbb{E}[g] \leq \varepsilon$, for $m = O(p/\varepsilon^2)$
- Overall: $\Pr(\|w' - w\|_p > 2\varepsilon) \leq e^{-\varepsilon^2 m}$ $\square$



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Approx. Carathéodory's Theorem

Given set V in the p-unit ball with norm $p \geq 2$, for every vector in the convex hull of V there exists an ε-close (under p-norm distance) vector that is a convex combination of $O\left(\frac{p}{\varepsilon^2}\right)$ vectors of V.

Khintchine-Kahane Inequality [TJ74,S11]

Let $r_1, r_2, \ldots, r_m$ be a sequence of i.i.d. random variables with $\Pr(r_i = \pm 1) = \frac{1}{2}$

In addition, let $u_1, u_2, \ldots, u_m \in \mathbb{R}^d$ be a deterministic sequence of vectors. Then, for $2 \leq p < \infty$

$$\mathbb{E}\left\| \sum_{i=1}^{m} r_i u_i \right\|_p \leq \sqrt{p}\ \left(\sum_{i=1}^{m} \|u_i\|_p^2 \right)^{\frac{1}{2}}$$