Conjectural Equilibrium in Water-filling Games

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***Abstract*—This paper considers a non-cooperative game in which competing users sharing a frequency-selective interference channel selfishly optimize their power allocation in order to improve their achievable rates. Previously, it was shown that a user having the knowledge of its opponents' channel state information can make foresighted decisions and substantially improve its performance compared with the case in which it deploys the conventional iterative water-filling algorithm, which does not exploit such knowledge. This paper discusses how a foresighted user can acquire this knowledge by modeling its experienced interference as a function of its own power allocation. To characterize the outcome of the multi-user interaction, the conjectural equilibrium is introduced, and the existence of this equilibrium for the investigated water-filling game is proved. Interestingly, both the Nash equilibrium and the Stackelberg equilibrium are shown to be special cases of the generalization of conjectural equilibrium. We develop practical algorithms to form accurate beliefs and search desirable power allocation strategies. Numerical simulations indicate that a foresighted user without any a priori knowledge of its competitors' private information can effectively learn the required information, and induce the entire system to an operating point that improves both its own achievable rate as well as the rates of the other participants in the water-filling game.**

Keywords-power control, game theory, conjectural equilibrium

I. INTRODUCTION

The multiuser power control problem in frequency-selective Gaussian interference channels was investigated from the game theoretic optimization perspective [1]-[11]. From a particular user's viewpoint, it is well-known that, for fixed interference, the optimal power allocation is the so-called water-filling solution. Therefore, the spectrum sharing problem can also be regarded as a *water-filling game*. Specifically, the participants in the water-filling game are modeled as players with individual goals and strategies. Existing research can be categorized into *non-cooperative* games and *cooperative* games.

First, the formulation of the multi-user environment as a non-cooperative game has appeared in several recent works [1]-[7]. An iterative water-filling (IW) algorithm has been proposed to mitigate the mutual interference and optimize the performance without the need for a central controller [1]. At every decision stage, selfish users maximize their achievable rates by water-filling across the entire band until a Nash equilibrium (NE) is reached. Sufficient conditions under which the iterative water-filling algorithm converges to a unique NE are derived and the closed form solution to the water-filling problem is investigated for some special scenarios [2]-[4].

On the other hand, because the IW algorithm may lead to Pareto-inefficient solutions [6], there also have been a number of related works studying spectrum sharing in cooperative games [8]-[11]. Several optimal algorithms were proposed to address the problem of weighted sum rate maximization. These works assume that users agree to cooperatively maximize a common objective and require explicit information exchanges among the users.

Our focus in this paper is on the non-cooperative setting, which considers the self-interested and competitive nature of individual players. However, most of the prior non-cooperative approaches often assume selfish users with only the knowledge of their private information and do not consider users' ability to improve their performance by exploring the information of the opponents. The best response strategy of a selfish user that knows its myopic opponents' private information, including their channel state information and power constraints, was first investigated in [7] using the Stackelberg equilibrium (SE) formulation in the two-user scenario. It was shown in [7] that surprisingly, a foresighted user playing the SE can improve both its performance as well as the performance of the other user. These results highlight the significance of information availability in water-filling games. However, one key question remains unsolved: how should a foresighted user acquire its desired information and adapt its response?

As opposed to our previous approach, which assumes a foresighted user with perfect knowledge of its competitors' private information [7], we discuss in this paper how the foresighted user without any such a priori knowledge can accumulate this knowledge and improve its performance. We propose that the foresighted user can explicitly model its competitors' response as a function of its power allocation by repeatedly interacting with the environment and observing the resulting interference. The concept of conjectural equilibrium (CE) is introduced to characterize the strategic behavior of a user that models the response of its myopic competing users, and the existence of this equilibrium in the water-filling game is proved. Some previously adopted solutions, including NE and SE, are shown to be special cases of the CE. Practical algorithms are developed to form accurate beliefs and search desirable power allocation strategy. It is shown that, a foresighted user can effectively learn its desired information and guide the entire system to an operating point having comparable performance to the algorithm in [7], where perfect a priori knowledge is assumed. More importantly, as opposed to the two-user algorithm in [7], the proposed algorithm can be applied in general scenarios where more than two users exist.

The rest of the paper is organized as follows. Section II presents the non-cooperative game model, reviews the existing non-cooperative solutions, and introduces the concept of CE. The existence of this CE in the water-filling game is proved in Section III. Section IV develops practical algorithms to form beliefs and approach the desired CE and provides the numerical results. Conclusions are drawn in Section V.

II. SYSTEM MODEL AND CONJECTURAL EQUILIBRIUM

A. System Description and Existing Solutions

We consider a frequency-selective interference channel. There are *K* transmitters and *K* receivers in the system. Each transmitter and receiver pair can be viewed as a player (or user). The entire frequency band is divided into a total number of *N* small frequency bins. The transfer function of the channel from transmitter *i* to receiver *j* in the *n*th channel is denoted as H_{ij}^n. Receiver *k*'s noise power spectral density (PSD) in the *n*th channel is denoted as N_k^n. Denote player *k*'s transmit PSD in the *n*th channel as P_k^n, which is subject to its power constraint:

$$\sum_{n=1}^{N} P_k^n \leq \mathbf{P_k^{max}}. \quad (1)$$

If interference is fixed and treated as noise, user *k* can achieve the following data rate (in nats):

$$R_k = \sum_{n=1}^{N} \ln\left[1 + P_k^n \Big/ \left(\sigma_k^n + \sum_{j\neq k} P_j^n \alpha_{jk}^n\right)\right]. \quad (2)$$

where $\sigma_k^n = N_k^n / |H_{kk}^n|^2$ and $\alpha_{jk}^n = |H_{jk}^n|^2 / |H_{kk}^n|^2$.

The multi-user interaction in the interference channel can be modeled as a game. Let $\mathcal{G} = (\mathcal{K}, \mathcal{A}, U)$ denote a game with $\mathcal{K} = \{1, \cdots, K\}$ being the set of players, $\mathcal{A} = \times_{k\in\mathcal{K}} \mathcal{A}_k$ being the set of actions available to the users (in which $\mathcal{A}_k$ is the set of actions available to user *k*), and $U = \times_{k\in\mathcal{K}} U_k$ being the users' payoff functions (in which $U_k : \mathcal{A} \to \mathcal{R}$ is the user *k*'s payoff function) [12]. In the water-filling game, users' actions are to select their transmit PSDs satisfying the constraint in (1) and the players' payoffs are their achievable data rates.

As mentioned earlier, existing research mainly focuses *cooperative* games and *non-cooperative* games. Specifically, cooperative approaches aim to maximize the weighted sum of data rates $\sum_{k=1}^{K} \omega_k R_k$ [8]-[10]. On the other hand, instead of solving the optimization problem globally, the non-cooperative IW algorithm models the users as myopic decision makers [1]. They optimize their transmit PSD by water-filling and compete to increase their transmission rates with the sole objective of maximizing their own data rates in (2). In other words, users are myopic, i.e., they update actions shortsightedly and do not consider the long-term impacts of taking these actions. The outcome of this non-cooperative scenario can be characterized by the concept of Nash equilibrium, which is defined to be any point $(a_1^*, \cdots, a_K^*)$ satisfying

$$U_k(a_k^*, a_{-k}^*) \geq U_k(a_k, a_{-k}^*) \text{ for all } a_k \in \mathcal{A}_k \text{ and } k \in \mathcal{K}, \quad (3)$$

where $a_{-k}^* = (a_1^*, \cdots, a_{k-1}^*, a_{k+1}^*, \cdots, a_K^*)$ [12]. The existence and the uniqueness of NE obtained by the IW algorithm are proved under a wide range of realistic conditions [2][3].

The recent approach in [7] demonstrates that the myopic behavior can be further improved because IW does not consider the coupling nature of players' actions and payoffs. If a selfish user gets the private information about its competitors and knows how they react, the best response strategy is to play the SE strategy. Let $NE(a_k)$ be the Nash equilibrium strategy of the remaining players if player k chooses to play a_k. The strategy profile $(a_k^*, NE(a_k^*))$ is a Stackelberg equilibrium with user k leading if and only if [13]

$$U_k(a_k^*, NE(a_k^*)) \geq U_k(a_k, NE(a_k)), \forall a_k \in \mathcal{A}_k. \quad (4)$$

Specifically, to find the SE in the water-filling game, we need to solve the following bi-level programming problem [7], where user 1 is assumed to be the foresighted user:

$$\begin{aligned}
&\max_{\{P_1^n\}} \sum_{n=1}^{N} \ln\left[1 + P_1^n \Big/ \left(\sigma_1^n + \sum_{k=2}^{K} \alpha_{k1}^n P_k^n\right)\right] \\
&s.t. \ \sum_{n=1}^{N} P_1^n \leq \mathrm{P}_{\mathbf{1}}^{\mathbf{max}}, \ and \ P_1^n \geq 0, \\
&P_k^n = \arg\max_{\{P_k'^n\}\in\mathcal{A}_k} \sum_{n=1}^{N} \ln\left[1 + P_k'^n \Big/ \left(\sigma_k^n + \sum_{i=1, i\neq k}^{K} \alpha_{ik}^n P_i^n\right)\right], \ k \geq 2
\end{aligned} \quad (5)$$

It should be pointed out that the foresighted user needs to know the private information $\{\sigma_k^n\}, \{\alpha_{ik}^n\}, \mathrm{P}_{\mathbf{k}}^{\mathbf{max}}$ of all its competitors in order to formulate the bi-level program. Previous approach in [7] assumes that the foresighted user has the perfect knowledge of this private information. Importantly, it was shown in [7] that users' performance is substantially improved compared with that of the IW algorithm if the foresighted user plays the SE strategy, even though the remaining users behave myopically. However, how such a foresighted user should accumulate this required information remains unsolved. In the remaining part of this paper, we will show that the foresighted user can obtain the information and improve its performance by forming conjectures over the behavior of its competitors through the repeated interaction.

B. Conjectural Equilibrium

In game-theoretic analysis, conclusions about the reached equilibria are based on assumptions about what knowledge the players possess. For example, the standard NE solution assumes that every player believes that the other players' actions will not change and chooses to myopically maximize its own payoff [12]. Hence, the players operating at equilibrium can be viewed as decision makers behaving optimally with respect to their *beliefs* about the policies of other players.

To rigorously define CE, we need to include two new elements $\mathcal{S}$ and s and, based on this, reformulate the strategic game $\mathcal{G} = (\mathcal{K}, \mathcal{A}, U, \mathcal{S}, s)$ [14]. $\mathcal{S} = \times_{k\in\mathcal{K}} \mathcal{S}_k$ is the *state space*, where $\mathcal{S}_k$ is the part of the state relevant to the *k*th user. Specifically, the state in the water-filling game is defined as the interference that users experience. The utility function $U = \times_{k\in\mathcal{K}} U_k$ is a map from users' state space and actions to real numbers, $U_k : \mathcal{S}_k \times \mathcal{A}_k \to \mathcal{R}$. The *state determination function* $s = \times_{k\in\mathcal{K}} s_k$ maps joint action to state with each component $s_k : \mathcal{A} \to \mathcal{S}_k$. Each user cannot directly observe the actions chosen by the others, and each user has some belief about the state that would result from performing its available actions. The *belief function* $\tilde{s} = \times_{k\in\mathcal{K}} \tilde{s}_k$ is defined to be $\tilde{s}_k : \mathcal{A}_k \to \mathcal{S}_k$

such that $\tilde{s}_k(a_k)$ represents the state that the player k believes that would result if it selects action a_k. Notice that the beliefs are not expressed in terms of other players' actions, and the multi-user coupling in these beliefs is captured directly by individual users forming conjectures of the effects of their own actions. In non-cooperative scenarios, each user chooses the action $a_k \in \mathcal{A}_k$ if it believes this action maximizes its utility.

Definition 1 **(*Conjectural Equilibrium*):** In game $\mathcal{G}$, a configuration of belief functions $(\tilde{s}_1^*, \cdots, \tilde{s}_K^*)$ and a joint action $a^* = (a_1^*, \cdots, a_K^*)$ constitute a CE, if for each $k \in \mathcal{K}$,

$$\tilde{s}_k^*(a_k^*) = s_k(a_1^*, \cdots, a_K^*) \text{ and } a_k^* = \arg\max_{a_k \in \mathcal{A}_k} U_k(\tilde{s}_k^*(a_k), a_k). \quad (6)$$

From the definition, we can see that, at CE, all users' expectations based on their beliefs are realized and each user behaves optimally according to its expectation. CE considers the users' beliefs rather than their perfect knowledge $NE(a_k)$ as in SE, which makes CE an appropriate solution when the perfect knowledge is not available. The key problem is how to configure the belief functions such that it leads to a CE having a satisfactory performance.

III. Conjectural Equilibrium in Water-filling Games

In this section, we propose to configure a user's belief about its interference as a linear function of its transmitted power, and show that such CE exists and it is a relaxation of both NE and SE. We begin by stating several fundamental assumptions used throughout the investigation hereafter.

i) User 1 is the only foresighted user modeling its competitors' reaction as a function of its own power allocation, and all the remaining users are myopic users that deploy the IW algorithm. For example, the foresighted user could be a secondary user in cognitive radio networks.

ii) Every user is able to perfectly measure its experienced noise PSD and interference PSD in all frequency channels.

iii) User $2, \cdots, K$ react to any small variation in their experienced interference by performing water-filling.

iv) A unique NE exists in the lower-level problem in (5). Sufficient conditions of the uniqueness of NE can be found in [2][3].

Next, we formally define the concept of stationary interference.

Definition 2 **(*Stationary Interference*):** The stationary interference that user 1 experiences in the *n*th channel is the accumulated interference $I_1^n = \sum_{i=2}^{K} \alpha_{i1}^n P_i^n$ when users 2,…,*K* reach their NE in the lower-level problem in (5). I_1^n is a function of user 1's power allocation $\boldsymbol{P}_1$ and it is also denoted as $I_1^n(\boldsymbol{P}_1)$.

A. Linear Belief of Stationary Interference

In the water-filling game, we define state $\mathcal{S}_k$ to be the stationary interference caused to user *k*, because besides its own power allocation, its utility also depends on the interference that its competitors cause to it. Note that user *k*'s action is to choose the transmitted power allocations subjected to its power constraint. By the definition of belief function, we need to express the stationary interference as a function of the transmitted power. As we will see later, deploying linear belief model captures the characteristics of the actual interference coupling and significantly improves the performance.

Define vector $\boldsymbol{\alpha^n} = \{\alpha_{ij}^n : i \neq j\}$, which contains the channel gains in the *n*th frequency bin. The following proposition motivates us to develop linear belief functions.

Proposition 1 **(*Linearity of Stationary Interference*):** If the number of frequency bins *N* is sufficiently large, the first derivative of the stationary interference that user 1 experiences in the *n*th channel with respect to its allocated power satisfies

$$\partial I_1^n / \partial P_1^n = c(\boldsymbol{\alpha^n}, P_1^n); \ \partial I_1^n / \partial P_1^m = 0, \ if \ m \neq n;$$

where $c(\boldsymbol{\alpha^n}, \boldsymbol{y})$ represents a constant determined by $\boldsymbol{\alpha^n}$ and $\boldsymbol{y}$.

Proof: By the definition of I_1^n, we have $\dfrac{\partial I_1^n}{\partial P_1^m} = \sum_{i=2}^{K} \alpha_{i1}^n \dfrac{\partial P_i^n}{\partial P_1^m}$. We only provide a sketch of the proof here. A detailed proof is provided in [16].

Without loss of generality, we assume that $P_k^n > 0$ for $k \in \{2, \cdots, K\}$. Let $NE_n(\boldsymbol{P}_1) = [P_2^n, \cdots, P_K^n]^T$ represent the power that user 2,…,*K* allocate in the *n*th channel at equilibrium. we have from the optimality conditions of water-filling solution: $(\boldsymbol{I} + \boldsymbol{G}) \cdot NE_n(\boldsymbol{P}_1) + \boldsymbol{g}^n P_1^n = \boldsymbol{\nu}$, in which

$$\boldsymbol{G} = \begin{bmatrix} 0 & \alpha_{32}^n & \alpha_{42}^n & \cdots & \alpha_{K2}^n \\ \alpha_{23}^n & 0 & \alpha_{43}^n & \cdots & \alpha_{K3}^n \\ \alpha_{24}^n & \alpha_{34}^n & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \alpha_{K,K-1}^n \\ \alpha_{2K}^n & \alpha_{3K}^n & \cdots & \alpha_{K-1,K}^n & 0 \end{bmatrix}, \boldsymbol{g}^n = \begin{bmatrix} \alpha_{12}^n \\ \alpha_{13}^n \\ \vdots \\ \alpha_{1K}^n \end{bmatrix}, \boldsymbol{\nu} = \begin{bmatrix} \nu_2 \\ \nu_3 \\ \vdots \\ \nu_K \end{bmatrix},$$

$\nu_i \ (i = 2, \cdots, K)$ are the water-levels of all the water-filling users. Note that the sufficient conditions of existence and uniqueness of NE generally require $\|\boldsymbol{G}\|_2 < 1$ [3], which leads to the fact that $\boldsymbol{I} + \boldsymbol{G}$ is invertible. Therefore, we have

$$NE_n(\boldsymbol{P}_1) = (\boldsymbol{I} + \boldsymbol{G})^{-1} \boldsymbol{\nu} - (\boldsymbol{I} + \boldsymbol{G})^{-1} \boldsymbol{g}^n P_1^n. \quad (7)$$

We also have $\lim_{N \to \infty} \partial \nu_i / \partial P_1^n = 0$, because if the width of each frequency bin F_s / N is sufficiently small, the fluctuation of the water-level is negligible. As a result, we have

$$\frac{\partial I_1^n}{\partial P_1^m} = \frac{\partial \boldsymbol{h}^n \cdot NE_n(\boldsymbol{P}_1)}{\partial P_1^m} = \begin{cases} -\boldsymbol{h}^n (\boldsymbol{I} + \boldsymbol{G})^{-1} \boldsymbol{g}^n, & if \ m = n \\ 0, & if \ m \neq n \end{cases}, \quad (8)$$

in which $\boldsymbol{h}^n = [\alpha_{21}^n \ \alpha_{31}^n \ \cdots \ \alpha_{K1}^n]$. ■

Proposition 1 indicates that, the first derivative with respect to user 1's allocated power in a certain channel is sufficient to capture how the stationary interference varies locally in that channel. We observe from Eq. (7) that $I_1^n = \boldsymbol{h}^n \cdot NE_n(\boldsymbol{P}_1) = \boldsymbol{h}^n (\boldsymbol{I} + \boldsymbol{G})^{-1} \boldsymbol{\nu} - \boldsymbol{h}^n (\boldsymbol{I} + \boldsymbol{G})^{-1} \boldsymbol{g}^n P_1^n$. Therefore, user 1 defines its belief function using the linear form $\tilde{I}_1^n = \beta^n - \gamma^n P_1^n$, in which γ^n is the estimate of $-\partial I_1^n / \partial P_1^n$ and β^n is a constant representing the composite effect of user 2,…,K's water-levels $\boldsymbol{\nu}$. This linear characterization of the stationary interference can greatly simplify the implicit functional expression $I_1^n(\boldsymbol{P}_1)$ given by the solution of lower-level problem (5) and maintain an accurate model of $I_1^n(\boldsymbol{P}_1)$ around the operating point $\boldsymbol{P}_1$.

B. *Existence of Conjectural Equilibrium*

Under the same known sufficient conditions discussed in [3] [7] for guaranteeing the existence of NE and SE, the existence of CE can be proved by showing that the first two types of equilibrium are special cases of CE. To this end, Table I compares the optimality conditions of the three types of equilibria in the water-filling game. It also shows that the information requirement for playing various equilibria differs. At NE, each user includes the interference I_k^n as a constant in its optimization and performs best response to I_k^n. To play SE, user 1 knows the functional expression of the stationary interference $I_1^n(\boldsymbol{P}_1')$ such that the bi-level program can be formed. Specifically, the required information includes the system-wide channel state information $\boldsymbol{\alpha^n}$, the noise PSD σ_k^n, and the individual power constraint $\mathrm{P}_{\mathbf{k}}^{\mathbf{max}}$ for any $n \in \{1,\cdots,N\}, k \in \mathcal{K}$. In contrast, in the case of CE, the above information is not required and user 1 behaves optimally with respect to its beliefs on how the stationary interference varies as a function of P_1^n.

***Proposition 2* (*NE & SE as CE*):** Both Nash equilibrium and Stackelberg equilibrium are special conjectural equilibria.

Proof: In order to show that both NE and SE are special cases of CE, we only need to verify that at NE and SE, user 1's action is optimal with respect to its belief and its belief agrees with its state. First, clearly, NE is a trivial CE with the parameters $\beta^n = \sum_{i=2}^{K} \alpha_{i1}^n P_i^n, \gamma^n = 0$ in user 1's belief functions. Next, denote $\boldsymbol{P}_{\mathrm{SE}} = \left[P_{SE}^1,\cdots,P_{SE}^N\right]$ the optimal solution of problem (5). To prove that SE is a CE, we need to find the corresponding β^n and γ^n, and show that SE solves the optimization problem:

$$\begin{gathered}\max \sum_{n=1}^{N} \log_2\left(1+\frac{P_1^n}{\sigma_1^n+\beta^n-\gamma^n P_1^n}\right) \\ s.t.\ \ P_1^n \geq 0\,, \beta^n-\gamma^n P_1^n \geq 0 \ and\ \sum\nolimits_{n=1}^{N} P_1^n \leq \mathbf{P}_1^{\mathbf{max}}.\end{gathered} \tag{9}$$

Consider the linear belief function with the parameters $\beta^n = \left(I_1^n - P_1^n \cdot \partial I_1^n / \partial P_1^n\right)\Big|_{\boldsymbol{P}_1=\boldsymbol{P}_{\mathrm{SE}}}$ and $\gamma^n = -\partial I_1^n / \partial P_1^n\Big|_{\boldsymbol{P}_1=\boldsymbol{P}_{\mathrm{SE}}}$. As discussed before, such parameters preserve all the local information of problem (5) around $\boldsymbol{P}_{\mathrm{SE}}$ into problem (9). KKT conditions hold at $\boldsymbol{P}_{\mathrm{SE}}$ since it solves problem (5). A sufficient condition which ensures SE to be a CE is that problem (9) belongs to convex optimization, because KKT conditions are necessary and sufficient for convex programming to attain its optimum. A sufficient condition SC **1** under which problem (9) is convex is that, for any $n \in \{1,\cdots,N\}$,

$$\text{SC }\mathbf{1}: \quad \left(I_1^n - P_1^n \cdot \partial I_1^n / \partial P_1^n\right)\Big|_{\boldsymbol{P}_1=\boldsymbol{P}_{\mathrm{SE}}} > 0 \ \ and$$

$$-\partial I_1^n / \partial P_1^n\Big|_{\boldsymbol{P}_1=\boldsymbol{P}_{\mathrm{SE}}} < \frac{1}{2} - \frac{1}{2\sigma_1^n} \cdot \left(I_1^n - P_1^n \cdot \partial I_1^n / \partial P_1^n\right)\Big|_{\boldsymbol{P}_1=\boldsymbol{P}_{\mathrm{SE}}}.$$

A detailed proof can be found in [16]. Therefore, we conclude that SE is also a special CE given these conditions. ■

Proposition 2 indicates that NE and SE are both CEs if parameters $\boldsymbol{\beta} = \{\beta^n\}, \boldsymbol{\gamma} = \{\gamma^n\}$ are properly chosen. Therefore, CE can be viewed as an operational approach to attain SE if the system-wide information required for solving SE is not available. It is because only the local information of stationary interference I_1^n and its first derivative $\partial I_1^n / \partial P_1^n$ is required to formulate problem (9), and this information can be obtained using measurements performed at the receiver.

In addition, we are interested in the existence of other CEs besides these two points. The following proposition indicates that infinite CEs may exist.

***Proposition 3* (*Infinite Set of CE*):** Suppose that all the users form conjectures according to Table I. The water-filling game may admit an infinite set of CE.

Proof: It can be proved using the maximum theorem and the implicit function theorem. Detailed proof is provided in [16].

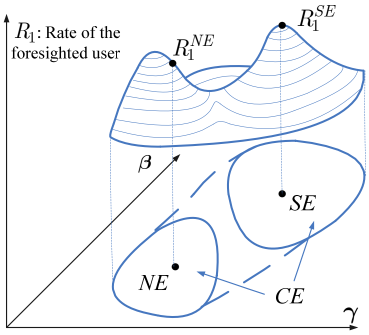

Fig. 1. Structure of conjectural equilibria in water-filling games.

In summary, proposition 1, 2, and 3 characterize the existence and structure of conjectural equilibrium in water-filling games. As shown in Fig. 1, NE and SE are both special CE. Open sets of CE that contain NE and SE may exist in the $\boldsymbol{\beta}$ - $\boldsymbol{\gamma}$ plane. SE attains the maximal data rate that a foresighted user can achieve. If the foresighted user properly sets up its parameters $\boldsymbol{\beta}, \boldsymbol{\gamma}$, the solution of CE in problem (9) coincides with the solution of SE in problem (5). More importantly, as opposed to the SE in which the knowledge of the system-wide private information is required, CE assumes that the foresighted user knows only its stationary interference and the first derivatives with respect to the allocated power, which greatly reduces the complexity of information acquisition.

IV. Achieving Desirable Conjectural Equilibria

Since proposition 3 shows that infinite CEs may exist and SE is the most desirable CE for a foresighted user, it should wisely choose the parameters β^n, γ^n of belief functions to attain SE as a CE. Moreover, the declarative conclusions drawn in Section III provide no hint on how to approach the CE. However, in practice, it is more important to construct algorithmic mechanisms to attain the desirable CE. This section proposes that users can update their beliefs in the repeated interaction setting and numerically examines their performance.

A. Conjecture-based Rate Maximization

Table II summarizes the dynamic updates of all users' states, belief functions, and optimal actions in the water-filling game. Specifically, at iteration t, users' states $I_{k,t}$ are determined by their opponents' power allocation. User 1 updates the parameters β_t^n, γ_t^n in its belief functions based on its state $I_{1,t}^n$ and allocated power $P_{1,t}^n$, and it also updates its power allocation $\boldsymbol{P}_{1,t+1}$ based on current operating points $\boldsymbol{P}_{1,t}$ and its belief $\tilde{\boldsymbol{I}}_{1,t}$. At the same time, myopic users $2,\cdots,K$ set their belief equal to their experienced interference and update their power allocation based on the water-filling strategy. Note that Table II implicitly assumes that user 1 will update after user $2,\cdots,K$'s IW algorithms converge such that user $2,\cdots,K$'s power allocations $\boldsymbol{P}_{k,t}$ at time t can be regarded as an equilibrium state. We can see from Table II that user 1 needs to complete two updates at each iteration. The entire procedure in Table II that enables the foresighted user to build beliefs and improve its performance is named "*Conjecture-based Rate Maximization*". Appropriate rules for updating beliefs are discussed as follows.

Update₁: β_t^n, γ_t^n

We have $I_1^n = \boldsymbol{h}^n(\boldsymbol{I}+\boldsymbol{G})^{-1}\boldsymbol{\nu} - \boldsymbol{h}^n(\boldsymbol{I}+\boldsymbol{G})^{-1}\boldsymbol{g}^n P_1^n$ from proposition 1, user 1's belief function takes the form of $\tilde{I}_1^n = \beta^n - \gamma^n P_1^n$, and it satisfies $I_1^n = \tilde{I}_1^n$ at CE for any $n \in \{1,\cdots,N\}$. As discussed in the previous section, by setting $\beta^n = I_1^n - P_1^n \cdot \partial I_1^n / \partial P_1^n$ and $\gamma^n = -\partial I_1^n / \partial P_1^n$, we can preserve all the local information of the original SE problem (5) around operating point $\boldsymbol{P}_{1,t}$. Therefore, we can update β_t^n and γ_t^n using $\beta_t^n = (I_1^n - P_1^n \cdot \partial I_1^n / \partial P_1^n)\big|_{\boldsymbol{P}_1=\boldsymbol{P}_{1,t}}$ and $\gamma_t^n = -\partial I_1^n / \partial P_1^n\big|_{\boldsymbol{P}_1=\boldsymbol{P}_{1,t}}$. User 1 can estimate the parameters using $\partial I_1^n / \partial P_1^n \approx \left[I_1^n\left(\{P_1^n+\varepsilon\}\cup \boldsymbol{P}_1^{-n}\right) - I_1^n\left(\{P_1^n-\varepsilon\}\cup \boldsymbol{P}_1^{-n}\right)\right]/2\varepsilon$ for small ε in which $\boldsymbol{P}_1^{-n} = \{P_1^1,\cdots,P_1^{n-1},P_1^{n+1},\cdots,P_1^N\}$.

After **Update₁** in each iteration, user 1 needs to solve problem (9). If proposition 2's assumption is not satisfied, problem (9) may belong to non-convex optimization, which is generally hard to solve. However, we are able to show that, as long as the number of frequency bins N is sufficiently large, problem (9) satisfies the time-sharing condition [9], which ensures the primal optimum is a convex function of the power constraint, and hence, the global optimum can be efficiently computed using the dual approach.

Proposition 4 (***Satisfaction of Time-sharing Condition***)**:** As the total number of sub-carriers N goes to infinity, problem (9) satisfies the time-sharing condition.

Proof: Due to space limits, the definition of time-sharing condition and the details of the proof are omitted. The reader is referred to [16].

Update₂: $\boldsymbol{P}_{1,t+1}$

It is shown in [9] that, if an optimization problem satisfies the time-sharing property, it has a zero duality gap, which leads to efficient dual algorithms that solve the non-convex problem. Consider the dual function

$$d(\eta) = \max_{\mathrm{P}_1'} \sum_{n=1}^{N}\left\{\log_2\left(1+\frac{P_1'^n}{\sigma_1^n+\beta_t^n-\gamma_t^n P_1'^n}\right) - \eta P_1'^n + \eta \mathbf{P}_1^{\max}\right\}.$$

Since $d(\eta)$ is convex, a bisection search in η converges to the global optimum. Specifically, Algorithm 1 summarizes such a dual method that solves problem (9) using bisection update. Hence, we can solve problem (9) regardless of its convexity.

Algorithm 1 : A dual method that solves problem (9) using bisection update

input: $\{\sigma_1^n\},\{\beta_t^n\},\{\gamma_t^n\},\mathbf{P}_1^{\max}$

initialization : $\boldsymbol{\eta}_{\min},\boldsymbol{\eta}_{\max},\boldsymbol{\eta}_0 = (\boldsymbol{\eta}_{\min}+\boldsymbol{\eta}_{\max})/2, i=0$

repeat

set $\boldsymbol{P}_1 = \left[P_1^1 \cdots P_1^N\right]$ where

$$P_1^n = \arg\max_{P_1'^n \ge 0 \text{ and } \gamma_t^n P_1'^n \le \beta_t^n} \log_2\left(1+\frac{P_1'^n}{\sigma_1^n+\beta_t^n-\gamma_t^n P_1'^n}\right) - \boldsymbol{\eta}_i P_1'^n \cdot$$

if $\sum_n P_1^n < \mathbf{P}_1^{\max}$, $\boldsymbol{\eta}_{\max} = \boldsymbol{\eta}_i$; else $\boldsymbol{\eta}_{\min} = \boldsymbol{\eta}_i$.

$\boldsymbol{\eta}_{i+1} \leftarrow (\boldsymbol{\eta}_{\min}+\boldsymbol{\eta}_{\max})/2,\ i=i+1\cdot$

until $\boldsymbol{\eta}_i$ converges

Table III. A dual algorithm that solves problem (9).

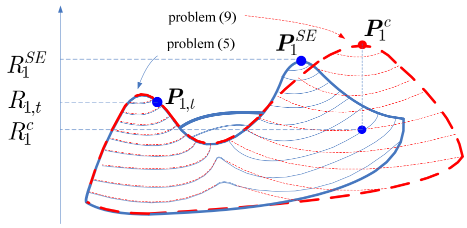

Fig. 2. Mismatch between problem (5) and (9).

Table IV summarizes the entire procedure of algorithm "Conjecture-based Rate Maximization" (CRM). We make several remarks about this algorithm. First, since we want to achieve better performance than NE, the initial operating point $\boldsymbol{P}_{1,0}$ is set to be the power allocation strategy $\boldsymbol{P}_1^{NE}$ that user 1 will choose if it adopts the IW algorithm. Second, in **Update₂**, the global optimum $\boldsymbol{P}_1^c$ is not directly used to update $P_{1,t+1}^n$. As shown in Fig. 2, this is because problem (9) is only a local approximation at $\boldsymbol{P}_{1,t}$ of the original SE problem (5) that we want to solve. Using $\boldsymbol{P}_1^c$ to update $\boldsymbol{P}_{1,t+1}$ may decrease the actual achievable rate R_1, if a mismatch between problem (5) and (9) exists for the solution $\boldsymbol{P}_1^c$. Therefore, **Update₂** adopts line search to improve the achievable rate. Third, as opposed to the two-user algorithm proposed in [7], CRM is designed for the general multi-user scenario regardless of the number of users. Last, CRM is not guaranteed to converge. It may stop with $v=0$ and $\boldsymbol{P}_{1,t} \ne \boldsymbol{P}_1^c$ in **Update₂**, i.e. the maximizer of

R_1 in the interval between $\boldsymbol{P}_{1,t}$ and $\boldsymbol{P}_1^c$ in Fig. 2 is $\boldsymbol{P}_{1,t}$. However, it can be verified that, since R_1 does not decrease, CRM terminates in limited iterations.

B. Illustrative Numerical Examples

We compare the performance of CRM with the IW algorithm and the two-user suboptimal algorithm (TSA) that searches SE assuming perfect knowledge of its opponent's private information [7]. We simulate a system with 200 sub-carriers over the 10-MHz band. We use a four-ray Rayleigh model with the exponential power profile and 100 ns root mean square delay spread.

	Required iterations				
	$t=1$	$t=2$	$t=3$	$t=4$	$t\geq 5$
CRM	0.59	0.07	0.26	0.07	0.01
Modified CRM	0.34	0.05	0.40	0.19	0.02

Table V. Iterations required by different CRM algorithms.

We first simulate the two-user scenario with $\mathrm{P}_1=\mathrm{P}_2=200$ and $\sigma_1^n=\sigma_2^n=0.01$. The total power of all rays of H_{11}^n and H_{22}^n is normalized as one, and that of H_{12}^n and H_{21}^n is normalized as 0.5. We tested 10^5 sets of frequency-selective fading channels where the Nash equilibrium exists. Denote user i's achievable rate using CRM, IW and TSA in [7] as R_i, R_i^{NE}, and $R_i^{\prime SE}$ respectively. Fig. 3 shows the simulated cumulative probability of the ratio of R_i over R_i^{NE} and $R_i^{\prime SE}$. The curve indicates that there is a probability of 59% that CRM returns the same power allocation as IW. The average improvement for user 1 of CRM over IW is 16.8%, which achieves almost the same performance as TSA. As shown in Fig. 3, $R_1/R_1^{\prime SE}$ is distributed symmetrically with respect to $R_1=R_1^{\prime SE}$. CRM provides for user 2 an average improvement of 20.7% over IW, which is smaller than TSA. In very few cases, CRM results in a rate $R_2^{\prime}$ smaller than R_2^{NE} in the IW algorithm.

The iteration time required by CRM is summarized in Table V. As mentioned above, CRM stops after just one iteration with a probability of 59% due to the problem mismatch shown in Fig. 2. In most scenarios, CRM terminates within 4 iterations and the average number of required iteration is only 1.84. To further improve the performance of CRM, we can modify the original CRM to handle the problem mismatch between (5) and (9). Notice that problem (9) is only a local approximation of problem (5) at $\boldsymbol{P}_{1,t}$. Additional constraints can be added in Algorithm 1, such that the optimum of problem (9) is searched only in a certain region around $\boldsymbol{P}_{1,t}$. For example, $|P_1^{\prime n}-P_{1,t}^n|$ can be restricted within a certain threshold when performing Algorithm 1 for any $n\in\{1,\cdots,N\}$. We simulated the two-user scenarios with additional restriction of $|P_1^{\prime n}-P_{1,t}^n|\leq 1$. Fig. 4 shows the simulated cumulative probability of R_i/R_i^{NE} for this modified CRM. As opposed to CRM, the probability that the modified CRM returns the same power allocation strategy as IW is reduced to 36% and the average performance improvement is also increased for both users. Specifically, the average performance improvement for user 1 is 24.4% and that of user 2 is 33.6%. However, Table V shows that the improvement is achieved at the cost of more iterations.

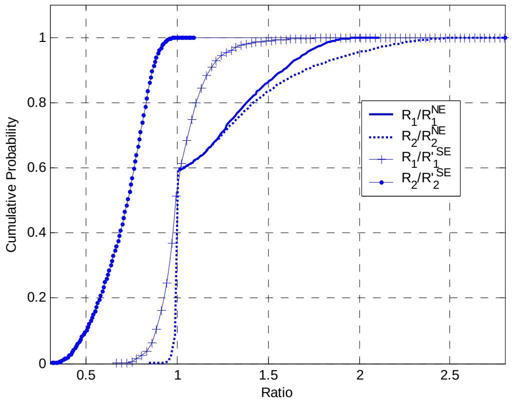

Fig. 3. Cdfs of R_i/R_i^{NE} and $R_i/R_i^{\prime SE}$ $(i=1,2)$.

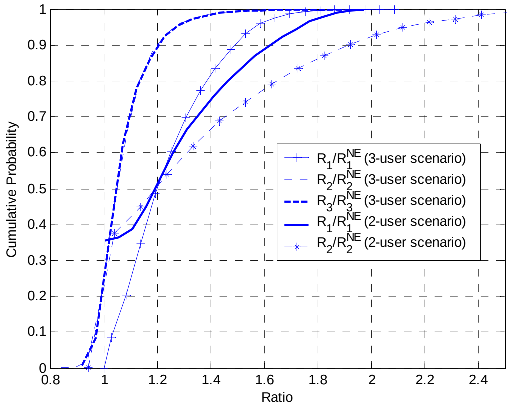

Fig. 4. Cdfs of R_i/R_i^{NE} for modified CRM.

We also tested performance of modified CRM in multi-user cases where TSA cannot be applied. We simulated the three-user scenarios with $\mathrm{P}_k=200$ and $\sigma_k^n=0.01$. The total power of all rays of H_{kk}^n is normalized as one, and that of H_{ij}^n $(i\neq j)$ is normalized as 0.33. Fig. 4 shows the simulated cdf of R_i/R_i^{NE}. The average improvement for user 1 of modified CRM over IW is 26.3%, and that of the rest users is 9.7%. We can see that, it benefits most of the participants in the water-filling game if a foresighted user forms accurate conjectures and plays the conjecture equilibrium strategy.

V. CONCLUSION

This paper introduces the concept of conjectural equilibrium in non-cooperative water-filling games and discusses how a user can model its experienced interference as a function of its own power allocation in order to improve its own data rate. The existence of conjectural equilibrium is proved, and both Nash equilibrium and Stackelberg equilibrium are shown to be special cases of conjectural equilibrium. Practical algorithms based on conjectural equilibrium are developed to determine desirable power allocation strategies. Numerical results verify that a user forming proper conjectures can improves both its own achievable rate as well as the rates of other participants, even if it has no a priori knowledge of its competitors' private information.

REFERENCES

[1] W. Yu, G. Ginis, and J. Cioffi, "Distributed multiuser power control for digital subscriber lines," *IEEE J. Sel. Areas Commun.* vol. 20, no. 5, pp.1105-1115, June 2002.

[2] Z. Luo and J. Pang, "Analysis of iterative waterfilling algorithm for multiuser power control in digital subscriber lines," *EURASIP Journal on Applied Signal Processing*, vol. 2006, Article ID 24012, 2006.

[3] K. W. Shum, K.-K. Leung, and C. W. Sung, "Convergence of iterative waterfilling algorithm for gaussian interference channels," *IEEE J. Sel. Areas Commun.*, vol. 25, no 6, pp. 1091-1100, Aug. 2007.

[4] E. Altman, K. Avrachenkov, and A. Garnaev, "Closed form solutions for symmetric water filling games," *Proc. IEEE Infocom*, pp. 673-681, 2008.

[5] R. Etkin, A. Parekh, and D. Tse, "Spectrum sharing for unlicensed bands," *IEEE J. Sel. Areas Commun.* vol. 25, no. 3, pp. 517-528, 2007.

[6] O. Popescu, D. Popescu, and C. Rose, "Simultaneous water filling in mutually interfering systems," *IEEE Trans. Wireless Commu.*, vol. 6, no. 3, pp. 1102-1113, Mar. 2007.

[7] Y. Su and M. van der Schaar, "A new look at multi-user power control games," *Proc. IEEE ICC 2008*, pp. 1072-1076, May 2008.

[8] R. Cendrillon, W. Yu, M. Moonen, J. Verlinden, and T. Bostoen, "Optimal multiuser spectrum balancing for digital subscriber lines," *IEEE Trans. Commu.*, vol. 54, no. 5, pp. 922-933, May 2006.

[9] W. Yu and R. Lui, "Dual methods for nonconvex spectrum optimization of multicarrier systems", *IEEE Trans. Commu.*, vol. 54, pp.1310-1322, 2006.

[10] R. Cendrillon, J. Huang, M. Chiang, and M. Moonen, "Autonomous spectrum balancing for digital subscriber lines," *IEEE Trans. on Signal Processing*, vol. 55, no. 8, pp. 4241-4257, Aug. 2007.

[11] J. Huang, R. Berry, and M. Honig, "Distributed interference compensation for wireless networks," *IEEE J. Sel. Areas Commun.*, vol. 24, no. 5, pp. 1074-1084, May 2006.

[12] D. Fudenberg and J. Tirole, *Game Theory*. Cambridge, MA, 1991.

[13] E. Altman, T. Boulogne, R. El-Azouzi, T Jimenez, and L. Wynter, "A survey on networking games," *Comp. Oper. Res.*, 2004.

[14] M. P. Wellman and J. Hu, "Conjectural equilibrium in multiagent learning," *Machine Learning*, vol. 33, pp. 179-200, 1998.

[15] C. Berge, *Topological Spaces*. New York: Dover, 1997.

[16] Y. Su and M. van der Schaar, "Conjectural Equilibrium in Water-filling Games," UCLA Technical Report, 2008 (arxiv.org/abs/0811.0048)

	User 1	User $2,\cdots,K$
Nash Equilibrium	$\{P_k^n\} = \arg\max_{\{P_k'^n\}\in\mathcal{A}_k} \sum_{n=1}^N \log_2\left(1+P_k'^n/(\sigma_k^n+I_k^n)\right)$	
Stackelberg Equilibrium	$\{P_1^n\} = \arg\max_{\{P_1'^n\}\in\mathcal{A}_1} \sum_{n=1}^N \log_2\left(1+P_1'^n/\left[\sigma_1^n+I_1^n(\boldsymbol{P}_1')\right]\right)$	$\{P_k^n\} = \arg\max_{\{P_k'^n\}\in\mathcal{A}_k} \sum_{n=1}^N \log_2\left(1+P_k'^n/\left[\sigma_k^n+I_k^n\right]\right)$
Conjectural Equilibrium	$\{P_k^n\} = \arg\max_{\{P_k'^n\}\in\mathcal{A}_k} \sum_{n=1}^N \log_2\left(1+P_k'^n/\left[\sigma_k^n+\tilde{I}_k^n\right]\right)$	
	$\tilde{I}_1^n = \beta^n - \gamma^n P_1^n, I_1^n = \sum_{i=2}^K \alpha_{i1}^n P_i^n, \tilde{I}_1^n = I_1^n$	$\tilde{I}_k^n = I_k^n = \sum_{i=1,i\neq k}^K \alpha_{ik}^n P_i^n$

Table I. Comparison among NE, SE, and CE in water-filling games.

	User 1	User $2,\cdots,K$
State $I_{k,t}$	$I_{k,t}^n = \sum_{i=1,i\neq k}^K \alpha_{i1}^n P_{i,t}^n$	
Belief function $\tilde{s}_k : \mathcal{A}_k \to \mathcal{S}_k$	$\beta_t^n, \gamma_t^n \leftarrow \mathbf{Update_1}\left(I_{1,t}^n, P_{1,t}^n\right), \tilde{I}_{1,t}^n = \beta_t^n - \gamma_t^n P_{1,t}^n$	$\tilde{I}_{k,t}^n = I_{k,t}^n = \sum_{i=1,i\neq k}^K \alpha_{ik}^n P_{i,t}^n$
Action $a_{1,t},\cdots,a_{K,t}$	$\boldsymbol{P}_{1,t+1} \leftarrow \mathbf{Update_2}\left(\boldsymbol{P}_{1,t}, \tilde{\boldsymbol{I}}_{1,t}\right)$	$\boldsymbol{P}_{k,t} = \arg\max_{\boldsymbol{P}_k' \in\mathcal{A}_k} \sum_{n=1}^N \log_2\left(1+P_k'^n/\left[\sigma_k^n+\tilde{I}_{k,t}^n\right]\right)$

Table II. Dynamic updates of the play.

Conjecture-based Rate Maximization

initialization : $t=0, \boldsymbol{P}_{1,0} = \boldsymbol{P}_1^{NE}$

repeat

I. $\beta_t^n, \gamma_t^n \leftarrow \mathbf{Update_1}\left(I_{1,t}^n, P_{1,t}^n\right)$.

II. $\boldsymbol{P}_{1,t+1} \leftarrow \mathbf{Update_2}\left(\boldsymbol{P}_{1,t}, \tilde{\boldsymbol{I}}_{1,t}\right)$, which includes:

1) Use Algorithm 1 to calculate the global optimum $\boldsymbol{P}_1^c$ of problem (9) with $\beta^n = \beta_t^n, \gamma^n = \gamma_t^n$.

2) Search in the interval $v\boldsymbol{P}_{1,t} + (1-v)\boldsymbol{P}_1^c \ (0\leq v\leq 1)$ and find the power allocation $\boldsymbol{P}_1^s$ maximizing user 1's rate R_1.

3) $\boldsymbol{P}_{1,t+1} \leftarrow \boldsymbol{P}_1^s, t = t+1$.

until no improvement can be made.

Table IV. Conjecture-based rate maximization.