Proofs of "Energy-Efficient Cooperative Communication using Selfish Relays"

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Theorem 1. *The sustainable strategies satisfy the following two properties:*

1) *The source agent strategy σ_s is $\sigma_s(k) = 0$, if $k = 0$;$\sigma_s(k) = 1$, if $k > 0$*
2) *The relay agent strategy σ_r is,$\exists K \in \mathbb{N}_+$, such that $\sigma_r(k) = 0$, if $k = 0$; $\sigma_r(k) = 1$, if $k > 0$ where K depends on c, b, β.*

Proof: (1) Suppose there is some k such that $\sigma_s(k) = 0$. If the source agent strategy is optimal, it implies that the marginal value of holding $k-1$ tokens is at least b/β, i.e., $V(k) - V(k-1) \geq b/\beta > b$. Consider any realized continuation history following the decision period. We estimate the loss in the expected utility having one less token. Because there is only one deviation in the initial time period, the following behaviors are exactly the same. The only difference occurs at the first time when the token holding drops to 0 when it is supposed to buy. At this moment, the agent cannot buy and losses benefit b. Therefore the loss in the utility is $\beta^t b$ for some t depending on the specific realized history. Because this analysis is valid for all possible histories, the expected utility is strictly less than b. This violates the optimality condition. Hence, it is always optimal for the source agent to spend the token if possible.

(2) We alternatively prove that the relay agent strategy cannot be non-threshold strategy in equilibrium in the following. The proof uses the results of Lemma 1.

For a non-threshold strategy, there must exist $K_1, K_2, (K_2 > K_1)$ such that

$$\begin{aligned} \sigma(k) = 1, &\quad 0 \leq k < K_1 \\ \sigma(k) = 0, &\quad K_1 \leq k < K_2 \\ \sigma(k) = 1, &\quad k = K_2 \end{aligned} \tag{1}$$

Following the same argument for a threshold strategy, we investigate the marginal utilities that below K_1. Suppose this strategy is an equilibrium, the following is true

$$M(0) > ... > M(K_1 - 1) \geq c/\beta, M(K_1) \leq c/\beta \tag{2}$$

It is also easy to see that

$$c/\beta \geq M(K_1) > M(K_1+1) > ... > M(K_2-2) > 0 \tag{3}$$

For the strategy to be an equilibrium, we need to check whether the following is also true.

$$M(K_2-1) \leq c/\beta, M(K_2) \geq c/\beta \tag{4}$$

However, we show that these two conditions cannot be satisfied at the same time. Specifically, we show that if $M(K_2) \geq c/\beta$, it must also be $M(K_2-1) > c/\beta$.

We discuss the following two cases:

Case 1: $\sigma(K_2+1) = 0$. For this case, we should have the following equation

$$\phi_l M(K_2-1) + \phi_c M(K_2) = (1-\mu)\rho c \tag{5}$$

Hence,

$$\begin{aligned}
&(1-\nu)\rho\beta M(K_2-1) \\
=&(1-\beta+(1-\nu+1-\mu)\rho\beta)M(K_2) - (1-\mu)\rho c \\
\geq&(1-\beta+(1-\nu+1-\mu)\rho\beta)\frac{c}{\beta} - (1-\mu)\rho c \\
=&(1-\beta)\frac{c}{\beta} + (1-\nu)\rho c > (1-\nu)\rho c
\end{aligned} \tag{6}$$

Therefore, $\sigma(K_2-1) > c/\beta$.

Case 2: $\sigma(K_2+1) = 1$. For this case, we should have the following equation

$$\phi_l M(K_2-1) + \phi_c M(K_2) + \phi_r M(K_2+1) = 0 \tag{7}$$

Also, there must exists $K_3 > K_2$ such that $\sigma(k) = 1, \forall K_2 \leq k < K_3$. ($K_3$ may also be infinite, we discuss this case later). Then following similar arguments for the threshold strategies, we have

$$M(K_2) > M(K_2+1) > ... > M(K_3-1) \geq c/\beta \tag{8}$$

With this result, we investigate (7)

$$\begin{aligned}
&(1-\nu)\rho\beta M(K_2-1) \\
=&(1-\beta+(1-\nu+1-\mu)\rho\beta)M(K_2) - (1-\mu)\rho\beta M(K_2+1) \\
>&(1-\beta)\frac{c}{\beta} + (1-\nu)\rho c > (1-\nu)\rho c
\end{aligned} \tag{9}$$

which also yields $\sigma(K_2-1) > c/\beta$.

Finally we discuss the case if $K_3 = +\infty$. For this case, we must have

$$M(k) > c/\beta, \forall k \geq K_2 \tag{10}$$

However, it contradicts the following condition that must be satisfied,

$$(\phi_l + \phi_c)(M(0) + M(K_2 - 1)) + (\phi_l + \phi_c + \phi_r) \sum_{k \neq 0, K_2 - 1} M(k) = (1 - \nu)\rho\beta \tag{11}$$

(Note $\phi_l + \phi_c > 0$, and $\phi_l + \phi_c + \phi_r > 0$.)

To sum up, a non-threshold strategy cannot be an equilibrium. ■

Lemma 1. *Fix b, c, β, the marginal utilities $M(k)$ of a threshold strategy has the following properties,*

1) $M(k) > 0, \forall k$
2) *$M(k)$ has no local maximum and at most one local minimum in $k \in \{1, ..., K-2\}$.*
3) *$M(k)$ is decreasing in $k \in \{K, K+1, ...\}$ and $\lim_{k \to +\infty} M(k) = 0$. Moreover, if the strategy is an equilibrium, $M(k)$ is decreasing in $k \in \{0, 1, ..., \}$.*

Proof: (1) For a threshold strategy σ_K, the utilities of having $k \in \{0, 1, ..., K\}$ tokens are given by,

$$\begin{aligned}
&V_\Pi(0) = \rho(1-\mu)(-c + \beta V_\Pi(1)) + (\rho(\mu+1) + 1 - 2\rho)\beta V_\Pi(0) \\
&V_\Pi(k) = \rho(1-\mu)(-c + \beta V_\Pi(k+1)) + \rho(1-\nu)(b + \beta V_\Pi(k-1)) + (\rho(\mu+\nu) + 1 - 2\rho)\beta V_\Pi(k), \\
&\qquad\qquad \forall k \in \{1, 2, \ldots, K-1\} \\
&V_\Pi(k) = \rho(1-\nu)(b + \beta V_\Pi(k-1)) + (\rho(\nu+1) + 1 - 2\rho)\beta V_\Pi(k), \forall k \in \{K, K+1, \ldots\}
\end{aligned} \tag{12}$$

After rearranging these equations and by introducing the auxiliary variables ϕ_l, ϕ_c, ϕ_r, we obtain,

$$\begin{aligned}
&(\phi_l + \phi_c)V_\Pi(0) + \phi_r V_\Pi(1) = -(1-\mu)\rho c \\
&\phi_l V_\Pi(k-1) + (\phi_l + \phi_c)V_\Pi(k) + \phi_r V_\Pi(k+1) = -(1-\mu)\rho + (1-\nu)\rho b, \forall k \in \{1, 2, ..., K-1\} \\
&\phi_l V_\Pi(k-1) + (\phi_l + \phi_c)V_\Pi(k) = (1-\nu)\rho b, \forall k \in \{K, K+1, \ldots\}
\end{aligned} \tag{13}$$

Consider $k = \in \{0, 1, \ldots, K\}$, we write the above equations in a more concise way by introducing a tridiagonal matrix $\tilde{\Phi}$,

$$\tilde{\Phi}\mathbf{V}_\Pi = \tilde{\mathbf{u}} \tag{14}$$

where

$$\tilde{\Phi} = \begin{bmatrix} \phi_l + \phi_c & \phi_r & 0 & \cdots & 0 \\ \phi_l & \phi_c & \phi_r & 0 & \vdots \\ 0 & \phi_l & \phi_c & \phi_r & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \phi_l & \phi_c + \phi_r \end{bmatrix}_{(K+1)\times(K+1)} \tag{15}$$

and $\tilde{\mathbf{u}} = [-(1-\mu)\rho c \quad -(1-\mu)\rho c + (1-\nu)\rho b \quad ... \quad -(1-\mu)\rho c + (1-\nu)\rho b \quad (1-\nu)\rho b]^T$.

Since,

$$\mathbf{V}_\Pi = \begin{bmatrix} 1 & 0 & \vdots & \vdots & 0 \\ 1 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 1 & & \cdots & 1 & 1 \end{bmatrix}_{(K+1)\times(K+1)} \begin{bmatrix} V_\Pi(0) \\ \mathbf{M}_\Pi \end{bmatrix} \tag{16}$$

We obtain,

$$\begin{bmatrix} \phi_l + \phi_c + \phi_r & \phi_r & 0 & \cdots & 0 \\ \phi_l + \phi_c + \phi_r & \phi_c + \phi_r & \phi_r & 0 & \vdots \\ \phi_l + \phi_c + \phi_r & \phi_l + \phi_c + \phi_r & \phi_c + \phi_r & \phi_r & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \phi_l + \phi_c + \phi_r & \cdots & \phi_l + \phi_c + \phi_r & \phi_l + \phi_c + \phi_r & \phi_c + \phi_r \end{bmatrix}_{(K+1)\times(K+1)} \begin{bmatrix} V_\Pi(0) \\ \mathbf{M}_\Pi \end{bmatrix} = \mathbf{u} \tag{17}$$

Subtracting the $(k-1)^{th}$ row from the k^{th} row, $\forall k \in \{2, ..., K+1\}$, we get

$$\begin{bmatrix} \phi_c & \phi_r & 0 & \cdots & 0 \\ \phi_l & \phi_c & \phi_r & 0 & \vdots \\ 0 & \phi_l & \phi_c & \phi_r & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \phi_l & \phi_c \end{bmatrix}_{K\times K} \begin{bmatrix} M_\Pi(0) \\ M_\Pi(1) \\ \vdots \\ M_\Pi(K-1) \end{bmatrix} = \begin{bmatrix} (1-\nu)\rho b \\ 0 \\ \vdots \\ 0 \\ (1-\mu)\rho c \end{bmatrix} \tag{18}$$

Denote it as

$$\Phi \mathbf{M}_\Pi = \mathbf{u} \tag{19}$$

Note $\phi_l < 0, \phi_c > 0, \phi_r < 0$, and it is also easy to see that

$$\phi_l + \phi_c + \phi_r > 0, \phi_l + \phi_c > 0, \phi_r + \phi_c > 0 \tag{20}$$

We first examine the sign of $M_\Pi(k), \forall k \in \{0, 1, ..., K-1\}$.

From (18), we obtain $\forall k \in \{1, 2, ..., K-2\}$,

$$\phi_l M_\Pi(k-1) + \phi_c M_\Pi(k) + \phi_r M_\Pi(k+1) = 0 \tag{21}$$

Suppose $\exists k^* \in \{1, 2, \ldots, K-2\}, M_\Pi(k^*) \leq 0$.

Case 1: If $0 \geq M_\Pi(k^*-1) \geq M_\Pi(k^*)$, since

$$M_\Pi(k^*+1) = \frac{\phi_l M_\Pi(k^*-1) + \phi_c M_\Pi(k^*)}{-\phi_r} \leq \frac{(\phi_l + \phi_c) M_\Pi(k^*)}{-\phi_r} \leq M_\Pi(k^*) \tag{22}$$

By iteration, $M_\Pi(K-1) \leq M_\Pi(K-2) \leq ... \leq M_\Pi(k^*) < M_\Pi(k^*-1) \leq 0$, which is not true because otherwise

$$\phi_c M_\Pi(K-1) = (1-\mu)\rho c - \phi_l M_\Pi(K-2) > -\phi_l M_\Pi(K-2) \geq -\phi_l M_\Pi(K-1) \tag{23}$$

and $\phi_c < -\phi_l$ which is a contradiction. With the same argument, $0 \geq M_\Pi(k^*) \geq M_\Pi(k^*+1)$ is not true either.

Case 2: If $0 \geq M_\Pi(k^*) \geq M_\Pi(k^*-1)$, similarly, since

$$M_\Pi(k^*-2) = \frac{\phi_r M_\Pi(k^*) + \phi_c M_\Pi(k^*-1)}{-\phi_l} \leq \frac{(\phi_r + \phi_c) M_\Pi(k^*-1)}{-\phi_l} \leq M_\Pi(k^*-1) \tag{24}$$

By iteration, $M_\Pi(0) \leq M_\Pi(1) \leq ... \leq M_\Pi(k^*-1) \leq M_\Pi(k^*) \leq 0$ which is not true either because otherwise

$$\phi_c M_\Pi(1) = \rho(1-\nu)b - \phi_r M_\Pi(2) > -\phi_r M_\Pi(2) \geq -\phi_r M_\Pi(1) \tag{25}$$

and $\phi_c < -\phi_r$ which is a contradiction. For the same reasoning, $0 \geq M_\Pi(k^*+1) \geq M_\Pi(k^*)$ is not true.

Therefore, neither $M_\Pi(k^*+1)$ nor $M_\Pi(k^*-1)$ can be non-positive. We show in the following that they cannot be positive either because otherwise

$$\phi_l M_\Pi(k^*-1) + \phi_c M_\Pi(k^*) + \phi_r M_\Pi(k^*+1) < 0 \tag{26}$$

Therefore $M_\Pi(k) > 0, \forall k \in \{1, ..., K-2\}$. Since

$$\phi_l M_\Pi(K-2) + \phi_c M_\Pi(K-1) = (1-\mu)\rho c > 0 \tag{27}$$

Hence, $M_\Pi(K-1) > 0$. Similarly $M_\Pi(0) > 0$. Now we conclude that $M_\Pi(k) > 0, \forall k \in \{0, 1, ..., K-1\}$.

For $k \in \{K, K+1, \ldots\}$, it is easy to show

$$\phi_l M_\Pi(k-1) + (\phi_c + \phi_r) M_\Pi(k) = 0 \tag{28}$$

hence,

$$M_\Pi(k) = \frac{-\phi_l}{\phi_c + \phi_r} M_\Pi(k-1) \tag{29}$$

since $M_\Pi(K-1) > 0$, recursively, $M(k) > 0, \forall k \in \{K, K+1, \ldots\}$.

(2) Firstly we show that there cannot be any local maximum in $M_\Pi(k)$ for $k = \{0, 1, ..., K-1\}$ except that on the boundary. We only need to check whether there exists some $k \in \{1, 2, ..., K-2\}$ such that

$$M_\Pi(k-1) \le M_\Pi(k) \ge M_\Pi(k+1) \tag{30}$$

This is not true, otherwise,

$$M_\Pi(k) = \frac{-\phi_l M_\Pi(k-1) - \phi_r M_\Pi(k+1)}{\phi_c} \le \frac{-\phi_l - \phi_r}{\phi_c} M_\Pi(k) < M_\Pi(k) \tag{31}$$

which is a contradiction.

Next we show that there is at most one local minimum in in $M_\Pi(k)$ for $k = \{0, 1, ..., K-1\}$ except that on the boundary. Suppose k^* is a local minimum.

$$M_\Pi(k^*-1) \ge M_\Pi(k^*) \le M_\Pi(k^*+1) \tag{32}$$

then

$$M_\Pi(k^*-2) = \frac{\phi_c M_\Pi(k^*-1) + \phi_r M_\Pi(k^*)}{-\phi_l} \ge \frac{\phi_c + \phi_r}{-\phi_l} M_\Pi(k^*-1) > M_\Pi(k^*-1) \tag{33}$$

and,

$$M_\Pi(k^*+2) = \frac{\phi_c M_\Pi(k^*+1) + \phi_l M_\Pi(k^*)}{-\phi_r} \ge \frac{\phi_c + \phi_l}{-\phi_r} M_\Pi(k^*+1) > M_\Pi(k^*+1) \tag{34}$$

By iteration, $M_\Pi(0) > M_\Pi(1) > ... > M_\Pi(k^*-1) \ge M_\Pi(k^*) \le M_\Pi(k^*+1) < ... < M_\Pi(K-2) < M_\Pi(K-1)$, meaning k is the only local minimum.

(3) It is easy to show $M_\Pi(k)$ is decreasing in $k \in \{K, K+1, \ldots\}$ by . Because $\frac{-\phi_l}{\phi_c+\phi_r}$ is a constant and strictly less than 1, $\lim_{k\to+\infty} M(k) = 0$.

Following we prove that in equilibrium, $M_\Pi(k)$ is decreasing in k. Suppose $M_\Pi(K-1) \ge M_\Pi(K-2)$, we prove it is not possible in equilibrium. Since in an equilibrium strategy,

$$M_\Pi(k) \ge \frac{c}{\beta}, \forall k = \{0, 1, ..., K-1\} \tag{35}$$

hence,

$$(1-\mu)\rho c = \phi_l M_\Pi(K-2) + \phi_c M_\Pi(K-1) \ge (\phi_l + \phi_c) M_\Pi(K-1) \ge (\phi_l + \phi_c)\tfrac{c}{\beta} > (1-\mu)\rho c \tag{36}$$

which is a contradiction. Hence, $M_\Pi(K-1) < M_\Pi(K-2)$. Since there is at most one local minimum and no local maximum in $k \in \{1, 2, \ldots, K-2\}$, we have $M_\Pi(k) > M_\Pi(k+1), \forall k \in \{0, 1, ..., K-2\}$, if Π is an equilibrium. Since $M_\Pi(k) > M_\Pi(k+1)$ is always true for $k \in \{K-1, K, \ldots\}$, the proof is completed.

■

Proposition 1. *Every threshold strategy admits a unique invariant token distribution. The invariant distribution is completely determined by threshold K and the token amount via the feasibility conditions (4) and the relations:*

$$\eta(k) = \left(\frac{1-\eta(0)}{1-\eta(K)}\right)^k \eta(0) \tag{37}$$

Proof: Suppose the token distribution at t is η_t. Following the given threshold strategy, the distribution evolves according to the following equations,

$$\eta_{t+1}(k) = \begin{cases} \rho(1-\eta_t(K))\eta_t(1) + [\rho(1+\eta_t(0)) + 1 - 2\rho]\eta_t(0), \text{if } k = 0 \\ \rho(1-\eta_t(0))\eta_t(k-1) + \rho(1-\eta_t(K))\eta_t(k+1) + [\rho(\eta_t(0)+\eta_t(K)) + 1 - 2\rho]\eta_t(k), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 < k < K \\ \rho(1-\eta_t(0))\eta_t(K-1) + [\rho(1+\eta_t(K)) + 1 - 2\rho]\eta_t(K), \text{if } k = K \end{cases} \tag{38}$$

The agents with k tokens at $t+1$ are consisted of three parts: agents with $(k-1)$ tokens at $t-1$ who get a token, agents with $(k+1)$ tokens at $t-1$ who lose a token and agents with k tokens at $t-1$ do not get or lose tokens. For an agent with $(k-1)$ tokens at time t, the conditions it receives one more token are (1) it is a server ;(2) its client has token to pay. Hence, the probability it receives a token is $\rho(1-\eta_t(0))$. Similarly, an agents with $(k+1)$ tokens, the conditions it loses a token are(1) it is a client ;(2) its server chooses to serve. Hence, the probability is $\rho(1-\eta_t(K))$. The conditions for an agent not to receive or lose tokens are (1) if it is a server, its client has no tokens; (2) if it is a client, its server refuses to serve (3) it is either a client or server. The probability is $\rho(\eta_t(0)+\eta_t(K)) + 1 - 2\rho$.

For an invariant distribution, $\eta_t(k) = \eta_{t+1}(k), \forall k \in \{0, 1, ..., K\}$. After some manipulations, we obtain

$$\eta(1) = \frac{1-\eta(0)}{1-\eta(K)}\eta(0) \tag{39}$$

and recursive equations

$$\eta(k) = \frac{2-\eta(0)-\eta(K)}{1-\eta(K)}\eta(k-1) - \frac{1-\eta(0)}{1-\eta(K)}\eta(k-2), \forall k = \{1, \ldots, K\} \tag{40}$$

Furthermore, we have

$$\eta(k) = \left(\frac{1-\eta(0)}{1-\eta(K)}\right)^k \eta(0) \tag{41}$$

Note it can be easily verified that $\sum_{k=0}^{K} \eta(k) = 1$ holds. ■

Proposition 2. *Given α, K, β, there exist $\gamma^L, \gamma^H (0 < \gamma^L < \gamma^H < 1)$ such that for all $\gamma \in [\gamma^L, \gamma^H]$, σ is a sustainable strategy; otherwise, it is not.*

Proof: Denote $r = 1/\gamma$. We alternatively prove:

Given α, K, β, there exist $r^L, r^H (1 < r^L < r^H)$ such that for all $r \in [r^L, r^H]$, σ is a sustainable strategy; otherwise, it is not.

We use the normalized marginal utility with respect to c in this proof.

(1) We first prove there exists r^L, such that $\forall r \geq r^L, M_\Pi(K-1) \geq 1/\beta$.

By Lemma 2, $F(r) = M_\Pi(K-1, r) - 1/\beta$ is an increasing function in r. We check the sign of $F(1)$ and $\lim_{r\to\infty} F(r)$ in the following.

Case 1: $r = 1$. Suppose $M_\Pi(K-1) \geq 1/\beta$, hence, $M_\Pi(k) \geq 1/\beta, \forall k \in \{0, 1, ..., K-1\}$. However, this is not true since

$$\begin{aligned}(1-\nu)\rho + (1-\mu)\rho &= (\phi_c + \phi_l)M_\Pi(0) + (\phi_c + \phi_l + \phi_r)\sum_{k=1}^{K-2} M_\Pi(0) + (\phi_c + \phi_r)M_\Pi(K-1) \\ &> K(1-\beta)/\beta + ((1-\nu) + (1-\mu))\rho > ((1-\nu) + (1-\mu))\rho\end{aligned} \tag{42}$$

Hence, $F(r) < 0$.

Case 2: $r \to \infty$. We prove $M_\Pi(K-1) \to \infty$. Suppose $M_\Pi(K-1)$ is upper-bounded, then $M_\Pi(k), \forall k \in \{0, 1, ..., K-1\}$ are also upper-bounded, assume $M_\Pi(k) < M < \infty, \forall k \in \{0, 1, ..., K-1\}$. However, this is not true because otherwise

$$\begin{aligned}(1-\nu)\rho r + (1-\mu)\rho &= (\phi_c + \phi_l)M_\Pi(0) + (\phi_c + \phi_l + \phi_r)\sum_{k=1}^{K-2} M_\Pi(0) + (\phi_c + \phi_r)M_\Pi(K-1) \\ &< K(1-\beta)M + ((1-\nu) + (1-\mu))\rho\beta M < \infty\end{aligned} \tag{43}$$

Hence, $\lim_{r\to\infty} F(r)$. Therefore, there exists an unique r^L such that for $r \geq r^L, M_\Pi(K-1) \geq 1/\beta$.

(2) Next we prove there exists $r^H > r^L$, such that $\forall r < r^H, M_\Pi(K-1) \leq \frac{\phi_c+\phi_r}{-\phi_l}\frac{1}{\beta}$. Denote $G(r) = M_\Pi(K-1, r) - \frac{\phi_c+\phi_r}{-\phi_l}\frac{1}{\beta}$, it is increasing in r. Check the sign of $G(r^L)$ and $\lim_{r\to\infty} G(r)$ in the following.

Case 1: $r = r^L$. It is easy to see $G(r^L) < 0$ since $\frac{\phi_c+\phi_r}{-\phi_l} > 1$.

Case 2: $r \to \infty$. It is also easy to see that $\lim_{r\to\infty} G(r) > 0$ with the same argument for $\lim_{r\to\infty} F(r)$. Therefore, there exists an unique r^H such that for $r \leq r^H, M_\Pi(K-1) \leq \frac{\phi_c+\phi_r}{-\phi_l}\frac{1}{\beta}$.

Combining (1) and (2), we get the result. ■

Lemma 2. *The normalized marginal utility $M(k)/c$ increases in the benefit-to-cost-ratio r, i.e., if $1 < r_1 < r_2$, then $\forall k$,*

$$\frac{M(k,r_1)}{c_1} < \frac{M(k,r_2)}{c_2} \tag{44}$$

Proof: Since

$$\Phi(\mathbf{M}_\Pi(r_2)/c_2 - \mathbf{M}_\Pi(r_1)/c_1) = \mathbf{u}_2/c_2 - \mathbf{u}_1/c_1 = (\rho(1-\nu)(r_2-r_1) \quad 0 \quad \dots \quad 0 \quad 0)^T \tag{45}$$

by Lemma 1 part (1),

$$M_\Pi(k,r_2)/c_2 - M_\Pi(k,r_1)/c_1 > 0, \forall k \in \mathbb{P} \tag{46}$$

■

Proposition 3. *Given α, K, γ, there exist $\beta^L, \beta^H (0 < \beta^L < \beta^H < 1)$ such that for all $\beta \in [\beta^L, \beta^H]$, σ is a sustainable strategy; otherwise, it is not.*

Proof: This proposition is similarly proved as Proposition 2. The proof uses the monotonicity of the marginal utility in β in Lemma 3. ■

Lemma 3. *The marginal utility $M(k,\beta$ increases in the discount factor β, i.e., if $0 \le \beta_1 \le \beta_2 < 1$, then $\forall k$,*

$$M(k,\beta_1) < M(k,\beta_2) \tag{47}$$

Proof: We prove this lemma by contradiction. Suppose $\exists k^*$ s.t. $M_\Pi(k^*,\beta_2) \le M_\Pi(k^*,\beta_1)$.

Let $\phi_l(\beta) = -\rho(1-\nu)\beta, \phi_c(\beta) = 1-(\rho(\mu+\nu)+1-2\rho)\beta, \phi_r(\beta) = -\rho(1-\mu)\beta$. Since $0 \le \beta_1 < \beta_2 < 1$,

$$0 < \frac{\phi_c(\beta_2)}{\beta_2} < \frac{\phi_c(\beta_1)}{\beta_1}, \qquad \frac{\phi_l(\beta_2)}{\beta_2} = \frac{\phi_l(\beta_1)}{\beta_1} < 0, \qquad \frac{\phi_r(\beta_2)}{\beta_2} = \frac{\phi_r(\beta_1)}{\beta_1} < 0. \tag{48}$$

$$\begin{gathered} 0 > \phi_l(\beta_1) > \phi_l(\beta_2), \quad 0 > \phi_r(\beta_1) > \phi_r(\beta_2), \quad \phi_c(\beta_1) > \phi_c(\beta_2) > 0, \\ \phi_l(\beta_1) + \phi_c(\beta_1) > \phi_l(\beta_2) + \phi_c(\beta_2) > 0, \quad \phi_r(\beta_1) + \phi_c(\beta_1) > \phi_r(\beta_2) + \phi_c(\beta_2) > 0, \\ \phi_l(\beta_1) + \phi_c(\beta_1) + \phi_r(\beta_1) > \phi_l(\beta_2) + \phi_c(\beta_2) + \phi_r(\beta_2) > 0 \end{gathered} \tag{49}$$

Since

$$\frac{\phi_l(\beta_1)}{\beta_1} M_\Pi(k^*-1,\beta_1) + \frac{\phi_c(\beta_1)}{\beta_1} M_\Pi(k^*,\beta_1) + \frac{\phi_r(\beta_1)}{\beta_1} M_\Pi(k^*+1,\beta_1) = 0 \tag{50}$$

$$\frac{\phi_l(\beta_2)}{\beta_2}M_\Pi(k^*-1,\beta_2)+\frac{\phi_c(\beta_2)}{\beta_2}M_\Pi(k^*,\beta_2)+\frac{\phi_r(\beta_2)}{\beta_2}M_\Pi(k^*+1,\beta_2)=0 \tag{51}$$

(50)-(51),

$$\begin{aligned}&-\tfrac{\phi_l(\beta_1)}{\beta_1}(M_\Pi(k^*-1,\beta_1)-M_\Pi(k^*-1,\beta_2))-\tfrac{\phi_r(\beta_1)}{\beta_1}(M_\Pi(k^*+1,\beta_1)-M_\Pi(k^*+1,\beta_2))\\&=\tfrac{\phi_c(\beta_1)}{\beta_1}M_\Pi(k^*,\beta_1)-\tfrac{\phi_c(\beta_2)}{\beta_2}M_\Pi(k^*,\beta_2)>0\end{aligned} \tag{52}$$

Hence, at least one of the following is true,

$$M_\Pi(k^*-1,\beta_1)-M_\Pi(k^*-1,\beta_2)>0 \quad\text{or}\quad M_\Pi(k^*+1,\beta_1)-M_\Pi(k^*+1,\beta_2)>0$$

Consider the case of $M_\Pi(k^*-1)>\hat{M}_\Pi(k^*-1)$, the other one is similar.

Summing up the $(k^*-1)^{th}$ and the k^{*th} row in (18) and divide by the discount factor, we obtain

$$\frac{\phi_l(\beta_1)}{\beta_1}M_\Pi(k^*-2,\beta_1)+\frac{\phi_l(\beta_1)+\phi_c(\beta_1)}{\beta_1}M_\Pi(k^*-1,\beta_1)+\frac{\phi_c(\beta_1)+\phi_r(\beta_1)}{\beta_1}M_\Pi(k^*,\beta_1)+\frac{\phi_r(\beta_1)}{\beta_1}M_\Pi(k^*+1,\beta_1)=0 \tag{53}$$

$$\frac{\phi_l(\beta_2)}{\beta_2}M_\Pi(k^*-2,\beta_2)+\frac{\phi_l(\beta_2)+\phi_c(\beta_2)}{\beta_2}M_\Pi(k^*-1,\beta_2)+\frac{\phi_c(\beta_2)+\phi_r(\beta_2)}{\beta_2}M_\Pi(k^*,\beta_2)+\frac{\phi_r(\beta_2)}{\beta_2}M_\Pi(k^*+1,\beta_2)=0 \tag{54}$$

(53)- (54),

$$\begin{aligned}&-\tfrac{\phi_l(\beta_1)}{\beta_1}(M_\Pi(k^*-2,\beta_1)-M_\Pi(k^*-2,\beta_2))-\tfrac{\phi_r(\beta_1)}{\beta_1}(M_\Pi(k^*+1,\beta_1)-M_\Pi(k^*+1,\beta_2))\\&=\tfrac{\phi_l(\beta_1)+\phi_c(\beta_1)}{\beta_1}M_\Pi(k^*-1,\beta_1)-\tfrac{\phi_l(\beta_2)+\phi_c(\beta_2)}{\beta_2}M_\Pi(k^*-1,\beta_2)\\&\quad+\tfrac{\phi_c(\beta_1)+\phi_r(\beta_1)}{\beta_1}M_\Pi(k^*,\beta_1)-\tfrac{\phi_c(\beta_2+\phi_r(\beta_2)}{\beta_2}M_\Pi(k^*,\beta_2)\\&>0\end{aligned} \tag{55}$$

Hence, at least one of the following is true,

$$M_\Pi(k^*-2,\beta_1)-M_\Pi(k^*-2,\beta_2)>0 \quad\text{or}\quad M_\Pi(k^*+1,\beta_1)-M_\Pi(k^*+1,\beta_2)>0$$

We continue this process by adding one more row from (18), either the upper one or the below one, we will eventually get to a place at least one of the following is true,

$$M_\Pi(k,\beta_1)>M_\Pi(k,\beta_2),\forall 0\le k\le k^* \quad\text{or}\quad M_\Pi(k,\beta_1)>M_\Pi(k,\beta_2),\forall k^*\le k\le K-1$$

Consider the first case is true, the other one is similar. Next we prove $M_{\Pi}(k^*+1, \beta_1) < M_{\Pi}(k^*+1, \beta_2)$ by summing up the first k^*+1 rows in (18),

$$\begin{aligned}
&-\phi_r(\beta_2)M_{\Pi}(k^*+1,\beta_2)+\rho(1-\nu)b \\
&=(\phi_l(\beta_2)+\phi_c(\beta_2))M_{\Pi}(0,\beta_2)+(\phi_l(\beta_2)+\phi_c(\beta_2)+\phi_r(\beta_2))\sum_{k=1}^{k^*-1}M_{\Pi}(k,\beta_2)+(\phi_c(\beta_2)+\phi_r(\beta_2))M_{\Pi}(k^*,\beta_2) \\
&<((\phi_l(\beta_1)+\phi_c(\beta_1))M_{\Pi}(0,\beta_1)+(\phi_l(\beta_1)+\phi_c(\beta_1)+\phi_r(\beta_1))\sum_{k=1}^{k^*-1}M_{\Pi}(k,\beta_1)+(\phi_c(\beta_1)+\phi_r(\beta_1))M_{\Pi}(k^*,\beta_1) \\
&=-\phi_r(\beta_1)M_{\Pi}(k^*+1,\beta_1)+\rho(1-\nu)b<-\phi_r(\beta_2)M_{\Pi}(k^*+1,\beta_1)+\rho(1-\nu)b
\end{aligned} \tag{56}$$

Hence, $M_{\Pi}(k^*+1, \beta_1) > M_{\Pi}(k^*+1, \beta_2)$. Iteratively, it is obvious to see $M_{\Pi}(k, \beta_1) > M_{\Pi}(k, \beta_2), \forall k \in \{0, 1, ..., K-1\}$. However, this is not true because, summing up all rows in (18)

$$\begin{aligned}
&\rho(1-\nu)b+\rho(1-\mu)c \\
&=(\phi_l(\beta_2)+\phi_c(\beta_2))M_{\Pi}(0,\beta_2)+(\phi_l(\beta_2)+\phi_c(\beta_2)+\phi_r(\beta_2))\sum_{k=1}^{K-2}M_{\Pi}(k,\beta_2)+(\phi_c(\beta_2)+\phi_r(\beta_2))M_{\Pi}(K-1,\beta_2) \\
&<((\phi_l(\beta_1)+\phi_c(\beta_1))M_{\Pi}(0,\beta_1)+(\phi_l(\beta_1)+\phi_c(\beta_1)+\phi_r(\beta_1))\sum_{k=1}^{K-2}M_{\Pi}(k,\beta_1)+(\phi_c(\beta_1)+\phi_r(\beta_1))M_{\Pi}(K-1,\beta_1) \\
&=\rho(1-\nu)b+\rho(1-\mu)c
\end{aligned} \tag{57}$$

which is a contradiction. Therefore, $M_{\Pi}(k, \beta_2) > M_{\Pi}(k, \beta_1), \forall k \in \{0, 1, \ldots, K-1\}$.

For $k \in \{K, K+1, \ldots\}$, if $M_{\Pi}(k-1, \beta_2) > M_{\Pi}(k-1, \beta_1)$,

$$\begin{aligned}
&M_{\Pi}(k,\beta_2)=\tfrac{-\phi_l(\beta_2)}{\phi_c(\beta_2)+\phi_r(\beta_2)}M_{\Pi}(k-1,\beta_2) \\
&>\tfrac{-\phi_l(\beta_1)}{\phi_c(\beta_1)+\phi_r(\beta_1)}M_{\Pi}(k-1,\beta_2)>\tfrac{-\phi_l(\beta_1)}{\phi_c(\beta_1)+\phi_r(\beta_1)}M_{\Pi}(k-1,\beta_1)=M_{\Pi}(k,\beta_1)
\end{aligned} \tag{58}$$

Since $M_{\Pi}(K-1, \beta_2) > M_{\Pi}(K-1, \beta_1)$, recursively, $M_{\Pi}(k, \beta_2) > M_{\Pi}(k, \beta_1), \forall k \in \{K, K+1, \ldots\}$. This completes the proof. ■

Theorem 2. *For each α, K, there is a non-empty set $\Phi(\alpha, K) = \{(\gamma, \beta) : (\sigma_K, \alpha) \text{is an equilibrium}\}$.*

1) *For fixed β, the γ section $\Phi_\beta(\alpha, K) = \{\gamma : (\gamma, \beta) \in \Phi(\alpha, K)\}$ is a non-empty closed interval.*
2) *For fixed γ, the β section $\Phi_\gamma(\alpha, K) = \{\beta : (\gamma, \beta) \in \Phi(\alpha, K)\}$ is a non-empty closed interval.*
3) *$\gamma^H(\beta), \gamma^L(\beta)$ are increasing in β, $\beta^H(\gamma), \beta^L(\gamma)$ are increasing in γ.*

Proof: This theorem is based on Proposition 2 and 3. The third bullet needs the overlap result in lemma 4. ■

Lemma 4. *Fix ρ, r, for two protocols $\Pi_1 = (\alpha_1 = K/2, \sigma_K)$ and $\Pi_2 = (\alpha_2 = (K+1)/2, \sigma_{K+1})$, the sustainable ranges of the discount factor are $[\beta_1^L, \beta_1^H]$ and $[\beta_2^L, \beta_2^H]$ respectively, then $\beta_2^L \in (\beta_1^L, \beta_1^H)$*

*and $\beta_1^H \in (\beta_2^L, \beta_2^H)$ **(the sustainable ranges of the discount factor overlap between two consecutive threshold protocols)***

Proof: For $\alpha_1 = K/2$, $\phi_l = \phi_r = -(1-\frac{1}{K+1})\rho\beta, \phi_c = 1-\beta+2\rho\beta(1-\frac{1}{K+1})$. We rewrite (18) as follows by dividing $(1-\frac{1}{K+1})$ on both sides of the equation, let $\tilde{\phi}_l = \tilde{\phi}_r = -\rho\beta$ and $\tilde{\phi}_c = 2\rho\beta + (1-\beta)(1+1/K)$,

$$\begin{bmatrix} \tilde{\phi}_c & \tilde{\phi}_l & 0 & \cdots & 0 \\ \tilde{\phi}_l & \tilde{\phi}_c & \tilde{\phi}_l & 0 & \vdots \\ 0 & \tilde{\phi}_l & \tilde{\phi}_c & \tilde{\phi}_l & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \tilde{\phi}_l & \tilde{\phi}_c \end{bmatrix}_{K\times K} \begin{bmatrix} M_{\Pi}(0) \\ M_{\Pi}(1) \\ \vdots \\ M_{\Pi}(K-1) \end{bmatrix} = \begin{bmatrix} \rho b \\ 0 \\ \vdots \\ 0 \\ \rho c \end{bmatrix} \tag{59}$$

(1) To prove $\beta_2^L > \beta_1^L$, equivalently we need to prove if $\beta = \beta_1^L$, the protocol Π_2 must have its threshold marginal utility $M_{\Pi_2}(K) < \frac{c}{\beta}$. We use contradiction to prove this. Suppose $M_{\Pi_2}(K) \geq \frac{c}{\beta}$.

$$M_{\Pi_2}(K) \geq M_{\Pi_1}(K-1) = \frac{c}{\beta} \tag{60}$$

Denote $\omega_K = (1/\beta - 1)(1+1/K)/\rho$, then,

$$M_{\Pi_1}(K-2) = (2+\omega_K)M_{\Pi_1}(K-1) - 1/\beta = (1+\omega_K)1/\beta \tag{61}$$

$$M_{\Pi_2}(K-1) = (2+\omega_{K+1})M_{\Pi_2}(K) - 1/\beta > (1+\omega_{K+1})1/\beta \tag{62}$$

Hence,

$$\frac{M_{\Pi_2}(K-1)}{M_{\Pi_1}(K-2)} > \frac{1+\omega_{K+1}}{1+\omega_K} > \frac{\omega_{K+1}}{\omega_K} = 1 - \frac{1}{(K+1)^2} \tag{63}$$

recursively, it is easy to show

$$\frac{M_{\Pi_2}(K-k)}{M_{\Pi_1}(K-k-1)} > \left(\frac{\omega_{K+1}}{\omega_K}\right)^k = \left(1 - \frac{1}{(K+1)^2}\right)^k, \forall k = 0, ..., K-1 \tag{64}$$

Using Taylor's expansion, we have

$$\left(1 - \frac{1}{(K+1)^2}\right)^k > 1 - \frac{k}{(K+1)^2} > 1 - \frac{1}{K+2}, \forall k = 0, ..., K-1 \tag{65}$$

Hence,

$$\frac{M_{\Pi_2}(K-k)}{M_{\Pi_1}(K-k-1)} > 1 - \frac{1}{K+2}, \forall k = 0, ..., K-1 \tag{66}$$

Next we prove $M_{\Pi_2}(0) \geq M_{\Pi_1}(0)$. Suppose this is not true, $M_{\Pi_2}(0) < M_{\Pi_1}(0)$, then it is easy to see by recursion that $M_{\Pi_2}(K-1) < M_{\Pi_1}(K-1) = \frac{c}{\beta}$, which contradicts that $M_{\Pi_2}(K-1) > M_{\Pi_2}(K) \geq \frac{c}{\beta}$.

Finally we consider the summation of all row equations in (59),

$$(1+\omega_{K+1})(M_{\Pi_2}(0)+M_{\Pi_2}(K))+\omega_{K+1}\sum_{k=1}^{K-1}M_{\Pi_2}(k)=b+c \tag{67}$$

$$(1+\omega_{K})(M_{\Pi_1}(0)+M_{\Pi_1}(K-1))+\omega_{K}\sum_{k=1}^{K-2}M_{\Pi_1}(k)=b+c \tag{68}$$

by Lemma 1,

$$M_{\Pi_2}(0)>M_{\Pi_2}(k),\forall k=1,...,K \tag{69}$$

hence,

$$\begin{aligned}
&b+c=M_{\Pi_2}(0)+M_{\Pi_2}(K)+\omega_{K+1}\sum_{k=0}^{K}M_{\Pi_2}(k)\\
&>M_{\Pi_2}(0)+M_{\Pi_2}(K)+\tfrac{K+1}{K}\omega_{K+1}\sum_{k=1}^{K}M_{\Pi_2}(k)\\
&>M_{\Pi_1}(0)+M_{\Pi_1}(K-1)+\tfrac{K+1}{K}\left(1-\tfrac{1}{(K+1)^2}\right)\left(1-\tfrac{1}{K+2}\right)\omega_K\sum_{k=0}^{K-1}M_{\Pi_1}(k)\\
&>M_{\Pi_1}(0)+M_{\Pi_1}(K-1)+\omega_K\sum_{k=0}^{K-1}M_{\Pi_1}(k)\\
&=b+c
\end{aligned} \tag{70}$$

This is a contradiction. Hence, $M_{\Pi_2}(K)<\frac{c}{\beta_1^L}$. Therefore $\beta_2^L>\beta_1^L$. In a similar fashion, $\beta_2^H>\beta_1^H$

(2) To prove $\beta_2^L<\beta_1^H$, equivalently we need to prove if $\beta=\beta_2^L$, the protocol Π_1 must have its threshold level marginal utility $M_{\Pi_1}(K-1)<\frac{\phi_c+\phi_r}{-\phi_l}\frac{c}{\beta}$.

For Π_2, when the discount factor is β_2^L, we have $M_{\Pi_2}(K)=c/\beta$. Substituting this into (59) for threshold $K+1$, we calculate $M_{\Pi_2}(K-1)=(1+(1/\beta-1)(1+1/(K+1))/\rho)c/\beta$. Then after removing the row and column that are related to $M_{\Pi_2}(K)$ of (59) for threshold $K+1$, we get a same size matrix equations with (59) for threshold K, and $M_{\Pi_2}(K-1)$ is determined as above.

We use contradiction to prove $M_{\Pi_1}(K-1)<\frac{\phi_c+\phi_r}{-\phi_l}\frac{c}{\beta}=(1+(1/\beta-1)(1+1/K)/\rho)c/\beta$. We this. Suppose $M_{\Pi_1}(K-1)\geq(1+(1/\beta-1)(1+1/K)/\rho)c/\beta$, then we have $M_{\Pi_1}(K-1)>M_{\Pi_2}(K-1)$.

Since

$$\rho\beta M_{\Pi_1}(K-2)=(2\rho\beta+(1-\beta)(1+1/K))M_{\Pi_1}(K-1)-\rho c \tag{71}$$

$$\rho\beta M_{\Pi_2}(K-2)=(2\rho\beta+(1-\beta)(1+1/(K+1)))M_{\Pi_2}(K-1)-\rho c \tag{72}$$

$M_{\Pi_1}(K-2)>M_{\Pi_2}(K-2)$. Recursively, it is easy to show $M_{\Pi_1}(k)>M_{\Pi_2}(k),\forall k=0,1,...,K-1$.

However, summing up all the rows equations in (59), we have

$$
\begin{aligned}
&\rho(b+c)\\
&=(\rho\beta+(1-\beta)(1+1/K))M_{\Pi_1}(0)+(1-\beta)(1+1/K)\sum_{k=1}^{K-2}M_{\Pi_1}(k)+(\rho\beta+(1-\beta)(1+1/K))M_{\Pi_1}(K-1)\\
&>(\rho\beta+(1-\beta)(1+1/(K+1)))M_{\Pi_2}(0)+(1-\beta)(1+1/(K+1))\sum_{k=1}^{K-2}M_{\Pi_2}(k)+\\
&\quad(\rho\beta+(1-\beta)(1+1/(K+1)))M_{\Pi_2}(K-1)\\
&=\rho(b+c)
\end{aligned}
\tag{73}
$$

A contradiction occurs. Hence, $M_{\Pi_1}(K-1) < \frac{\phi_c+\phi_r}{-\phi_l}\frac{c}{\beta_2^L} = (1+(1/\beta_2^L-1)(1+1/K)/\rho)c/\beta_2^L$, therefore $\beta_2^L < \beta_1^H$.

Combining (1) and (2), we complete the proof. ∎

Proposition 4. *For the threshold strategy σ_K with $K<+\infty$, the cooperation probability is maximized when $\alpha = K/2$ in which case the cooperation probability is*

$$\max R(P_r^{AF},\sigma_K,\alpha) = \left(\frac{K}{K+1}\right)^2 \tag{74}$$

Proof: Consider the following maximization problem

$$
\begin{aligned}
\underset{0\le x_1,x_2\le 1}{\text{maximize}} \quad & E^*(x_1,x_2) = 1-x_1-x_2+x_1x_2\\
\text{subject to} \quad & x_1(1-x_1)^K = x_2(1-x_2)^K
\end{aligned}
\tag{75}
$$

We find the optimal solution in the following. Consider $f(x)=x(1-x)^K$, we show that $\forall 0\le x_1\le \frac{1}{K+1}\le x_2\le 1$ such that $f(x_1)=f(x_2)$, the following is true.

1) $x_1+x_2 \ge \frac{2}{K+1}$, with equality achieved at $x_1=x_2=\frac{1}{K+1}$.

2) $x_1x_2 \le \frac{1}{K+1}$, with equality achieved at $x_1=x_2=\frac{1}{K+1}$.

They are proved in the following.

(1) We first show $f(x)$ is increasing in $[0,\frac{1}{K+1}]$ and decreasing in $[\frac{1}{K+1},1]$. It is sufficient to check the first order condition,

$$\frac{df(x)}{dx} = (1-x)^K - Kx(1-x)^{K-1} = (1-x)^{K-1}(1-x-Kx) \tag{76}$$

Let $\Delta = x-\frac{1}{K+1}$. Hence, $g(\Delta)=f(\frac{1}{K+1}+\Delta)=(\frac{1}{K+1}+\Delta)(1-\frac{1}{K+1}-\Delta)^K$.

$$\frac{dg(\Delta)}{d\Delta} = -(K+1)(1-\frac{1}{K+1}-\Delta)^{K-1}\Delta \tag{77}$$

Therefore, $\forall\Delta\in[0,\frac{1}{K+1}]$,

$$|g'(\Delta)|-|g'(-\Delta)|\le 0 \tag{78}$$

Or equivalently

$$|f'(\frac{1}{K+1}+\Delta)|-|f'(\frac{1}{K+1}-\Delta)|\leq 0 \tag{79}$$

with equality achieved only at $\Delta = 0$.

This implies that the absolute value of the slope of $x \in [\frac{1}{K+1}, 1]$ is slower than that of $x \in [0, \frac{1}{K+1}]$. Hence, for $\Delta = \frac{1}{K+1} - x_1$, we have

$$f(x_2) = f(x_1) \leq f(\frac{2}{K+1} - x_1) \tag{80}$$

Since, $f(x)$ is decreasing in $[\frac{1}{K+1}, 1]$,

$$x_2 \geq \frac{2}{K+1} - x_1 \tag{81}$$

Therefore,

$$x_1 + x_2 \geq \frac{2}{K+1} \tag{82}$$

with equality achieved only at $x_1 = x_2 = \frac{1}{K+1}$

(2) Denote $h(x_1, x_2) = x_1 x_2$, we need to check the first order condition of $h(x_1, x_2)$ with respect to x_1. Before doing that, we calculate the following,

$$\frac{\partial x_2}{\partial x_1} = \frac{(1-x_1)^{K-1}(1-x_1-Kx_1)}{(1-x_2)^{K-1}(1-x_2-Kx_2)} \tag{83}$$

Then we have

$$\frac{\partial h}{\partial x_1} = x_2 + x_1\frac{\partial x_2}{\partial x_1} = \frac{(1-x_1)(x_2+Kx_2-1)-(1-x_2)(1-x_1-Kx_1)}{(1-x_1)(x_2+Kx_2-1)}x_2 \geq 0 \tag{84}$$

with equality achieved only at $x_1 = x_2 = \frac{1}{K+1}$. The inequality is due to $1-x_1 \geq 1-x_2$ and $(1+K)x_2 - 1 \geq 1-(1+K)x_1$. Therefore, $h(x_1, x_2) \leq \left(\frac{1}{K+1}\right)^2$ with equality achieved only at $x_1 = x_2 = \frac{1}{K+1}$.

Combining (1) and (2), we obtain $\max E^*(x_1, x_2) = \left(1 - \frac{1}{K+1}\right)^2$ with the optimal solution $x_1 = x_2 = \frac{1}{K+1}$. Also note $\max E^*(x_1, x_2) \geq \max E(\Pi)$ due to the relaxed constraints, hence we have $\max E(\Pi) \leq \left(1-\frac{1}{K+1}\right)^2$. However, $\eta(0) = \eta(1) = \ldots = \eta(K) = \frac{1}{K+1}$ is a feasible solution that achieves the optimal value. Also, α is easily calculated as $K/2$.

In sum, the efficiency E is maximized at $\alpha = K/2$, and $E^{opt}(K) = E(K/2, \sigma_K) = \left(1 - \frac{1}{K+1}\right)^2$.

■

Theorem 3. 1) *R^{opt} is increasing in β and decreasing in γ;*

2) $\lim_{\beta \to 1} R^{opt}(\gamma, \beta) = 1$*;*

3) $\lim_{\gamma \to 0} R^{opt}(\gamma, \beta) = 1$

Proof: The results are mainly based on Theorem 2 and Lemma 4. ■

Proposition 5. *$\forall \beta, \gamma$, if α is upper-bounded $\alpha \leq \alpha_0$, then the optimal possible cooperation probability is bounded away from 1 no matter how large the threshold is, in particular,*

$$R < 1 - \frac{1}{2\lceil \alpha_0 \rceil + 1} \tag{85}$$

Proof: Let $K^\star = 2\lceil \alpha_0 \rceil$. For any protocol $\Pi = (\alpha, K), K \leq K^\star$, the efficiency $E \leq \left(1 - \frac{1}{K^\star+1}\right)^2 = \left(1 - \frac{1}{2\lceil \alpha_0 \rceil+1}\right)^2 < \left(1 - \frac{1}{2\lceil \alpha_0 \rceil+1}\right)$.

Now we consider the protocol $\Pi' = (\alpha, K'), K' > K^\star$, we prove $\eta'(0) \geq \eta^\star(0)$.

Firstly, $\eta^\star(0) \geq \frac{1}{K^\star+1}$. This is easily seen because for the protocol $\Pi = (K^\star/2, K^\star)$, the token distribution is uniformly distributed $\eta(k) = 1/(K^\star + 1)$. Since $\alpha \leq K^\star/2$, this will increase the number of agents who have no tokens, i.e. $\eta^\star(0) \geq \frac{1}{K^\star+1}$.

Then we consider $\Pi' = (K^\star/2, K'), K' > K^\star$.

(1) Suppose $\exists k$, such that $\eta'(k) \geq 1/(K^\star + 1)$, then by the monotonicity of the token distribution, $\eta'(0) \geq 1/(K^\star + 1)$.

(2) Suppose $\forall k \in \{0, 1, ..., K'\}, \eta'(k) < 1/(K^\star + 1)$, we compare with the distribution of $\Pi^\star = (K^\star/2, K^\star)$, i.e. $\eta^\star(k) = \frac{1}{K^\star+1}, \forall k \in \{0, 1, ..., K^\star\}$. Since $\forall k \in \{0, 1, ..., K^\star\}, \eta'(k) < \eta^\star(k)$, and $\sum\limits_{k=K^\star+1}^{K'} \eta'(k) < \sum\limits_{k=0}^{K^\star} (\eta^\star(k) - \eta'(k))$, hence, $\sum\limits_{k=0}^{K'} \eta'(k) < \sum\limits_{k=0}^{K^\star} \eta^\star(k) = 1$. This is an contradiction, hence, $\exists k$, such that $\eta'(k) \geq 1/(K^\star + 1)$. Therefore $\eta'(0) \geq 1/(K^\star + 1)$.

Therefore, $K' > K^\star$, the efficiency is also bounded away from 1, $E = (1 - \eta'(0))(1 - \eta'(K)) < (1 - \eta'(0)) < (1 - 1/(K^\star + 1)) = \left(1 - \frac{1}{2\lceil \alpha_0 \rceil+1}\right)$. The proof is now completed.

■

References

[1] Y. Zhang and M. van der Schaar, "Social Norms for Online Communities", *UCLA technical report*, 2011.