Towards a Theory of Societal Co-Evolution: Individualism versus Collectivism

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I. Appendix

Theorem 1: Every society has a unique steady state.

Proof: We will start by deriving the population density at a given welfare level x, $p_{\lambda_b,\lambda_d,r,w}(x)$ and the total population mass $Pop(\lambda_d,\lambda_b,r,w)$ in the steady state and show that they are unique. To do so we first arrive at the expression for the normalized population density $f_{\lambda_b,\lambda_d,r,w}(x)$. The relation between $f_{\lambda_b,\lambda_d,r,w}(x)$, $p_{\lambda_b,\lambda_d,r,w}(x)$ and $Pop(\lambda_d,\lambda_b,r,w)$ is given as, $Pop(\lambda_d,\lambda_b,r,w) = \int_{-\infty}^{\infty} p_{\lambda_b,\lambda_d,r,w}(x)dx$, $f_{\lambda_b,\lambda_d,r,w}(x) = \frac{p_{\lambda_b,\lambda_d,r,w}(x)}{Pop(\lambda_d,\lambda_b,r,w)}$. In steady state the average impact of the society, i.e. $\bar{Q}(\lambda_b,\lambda_d,r,w)$ is determined since the proportion of individuals with $Q=q$, i.e. $M(Q=q)$ do not change. Hence, the rate at which the welfare of an individual grows can take only two values depending on his quality, $R_1 = (1-w).1 + w.\bar{Q}(\lambda_b,\lambda_d,r,w)$, $R_{-1} = (1-w).-1 + w.\bar{Q}(\lambda_b,\lambda_d,r,w)$, here R_1 and R_{-1} are the rate of growth of good and bad quality individual respectively. To derive the densities in steady state, we will first show that in the steady state R_1 and R_{-1} will be positive and negative respectively. Let's assume that R_1 and R_{-1} are both positive, i.e. all the individuals in the society experience a positive growth. In such a case the individuals can only die due to a Poisson arrival. Also, we know that an individual who is born is as likely to be good as he is to be bad. Hence, the population mass at which the rate of death will equal the rate of birth of good/bad quality individual is the same for both the types of individuals, i.e. $M(Q=+1) = M(Q=-1)$. As a result, the average quality $\bar{Q}(\lambda_b,\lambda_d,r,w) = 0$. Substituting this back in the expressions for the rate we get, $R_1 = (1-w)$ and $R_{-1} = (1-w).-1$. Therefore, R_{-1} is negative this contradicts the supposition that the both the rates are positive. Next, let's assume that both R_1 and R_{-1} are negative. In this case the individuals can die either due to a Poisson arrival or due to hitting the death boundary. In such a case the welfare values attained will only be negative. Let $f^{1}_{\lambda_b,\lambda_d,r,w}(x)$ correspond to the joint density that the individual of good quality attains a welfare level of x. Similarly, we can define $f^{-1}_{\lambda_b,\lambda_d,r,w}(x)$ to be the joint density for a bad quality individual at a given welfare level of x. In steady state although the density of population in a given welfare level is fixed, however the individuals comprising the density at a given welfare level is not the same owing to change of welfare levels, births and deaths that happen continually. As a result, at any instant of time the mass of individuals that attain a given welfare level will equal the mass of indiduals that leave that welfare level either due to change in welfare or due to dying. Consider an infinitesimal interval h, the mass of the population with quality $Q=1$ between

$x-h$ and x at time t, where $x \leq 0$, is given as, $f^{1}_{\lambda_b,\lambda_d,r,w}(x).h$. Consider a time interval $t^{'}$ after which this mass of individuals, $f^{1}_{\lambda_b,\lambda_d,r,w}(x).h$ will either die or will attain a different welfare level between, $y-h$ and y, here $y = x + R_1.t^{'}$. The probability that an individual does not die a natural death in time interval $t^{'}$ is $e^{-\lambda_d t^{'}}$. Hence, the proportion of the mass of individuals who do not die a natural death and a result attain a welfare between $y-h$ and y is $e^{-\lambda t^{'}} f^{1}_{\lambda_b,\lambda_d,r,w}(x).h = f^{1}_{\lambda_b,\lambda_d,r,w}(y).h$. This can be expressed as $f^{1}_{\lambda_b,\lambda_d,r,w}(y) = e^{-\lambda_d \frac{y-x}{R_1}} f^{1}_{\lambda_b,\lambda_d,r,w}(x)$ and $f^{1}_{\lambda_b,\lambda_d,r,w}(y) = C_1.e^{-\lambda_d \frac{y}{R_1}}$ where $f^{1}_{\lambda_b,\lambda_d,r,w}(0) = C_1$. Similarly, for $y \leq 0$ we can get $f^{-1}_{\lambda_b,\lambda_d,r,w}(y) = C_{-1}.e^{\lambda_d \frac{y}{R_{-1}}}$ where $f^{-1}_{\lambda_b,\lambda_d,r,w}(0) = C_{-1}$. Note that both $f^{1}_{\lambda_b,\lambda_d,r,w}(x)$ and $f^{-1}_{\lambda_b,\lambda_d,r,w}(x)$ are zero for positive welfare values since both good and bad quality individuals are assumed to have a negative rate of growth. Also, the rate at which individuals of good quality and bad quality are born is the same given as $\frac{\lambda_b}{2}$. Hence, we can equate the mass of good (bad) quality individuals which enter the society in time δt, i.e. $\frac{\lambda_b}{2}\delta t$ to the mass of individuals between welfare level of 0 and δx_1 (0 and δx_2), i.e. $C_1 \delta x_1$ ($C_{-1}\delta x_{-1}$). This gives, $C_{-1}R_{-1} = C_1 R_1 = C$. Since the $f^{1}_{\lambda_b,\lambda_d,r,w}(x)$ and $f^{-1}_{\lambda_b,\lambda_d,r,w}(x)$ are joint density functions the integral of the sum of these joint densities should be 1.

$$\begin{aligned}
\int_{-\infty}^{\infty} f^{1}_{\lambda_b,\lambda_d,r,w}(x)dx + \int_{-\infty}^{\infty} f^{-1}_{\lambda_b,\lambda_d,r,w}(x)dx &= 1 \\
\frac{C_1 R_1}{\lambda_d}(1 - e^{\frac{\lambda_d}{R_1}r}) + \frac{C_{-1}R_{-1}}{\lambda_d}(1 - e^{\frac{\lambda_d}{R_{-1}}r}) &= 1 \\
C = \frac{\lambda_d}{2 - e^{\frac{\lambda_d}{R_1}r} - e^{\frac{\lambda_d}{R_{-1}}r}}
\end{aligned}$$

From this we can calculate the mass of the individuals with $Q = 1$ and $Q = -1$, i.e. $M(Q = +1) = \frac{C}{\lambda_d}(1 - e^{\frac{\lambda_d}{R_1}r})$ and $M(Q = -1) = \frac{C}{\lambda_d}(1 - e^{\frac{\lambda_d}{R_{-1}}r})$. Since $R_1 > R_{-1}$ we can see that $M(Q = +1) > M(Q = -1)$. This yields that the $\bar{Q}(\lambda_b, \lambda_d, r, w) > 0$ and thereby $R_1 > 0$. This contradicts the supposition that both the rates are negative. Also, since $R_1 > R_{-1}$ the only case left is R_1 is positive while R_{-1} is negative. In this case the good and bad quality individuals take positive and negative welfare values respectively. We can calculate the joint densities in the same manner as described above and thus the resulting density is $f^{1}_{\lambda_b,\lambda_d,r,w}(x) = C_1 e^{-\frac{\lambda_d}{R_1}x}$, $x > 0$ and $f^{-1}_{\lambda_b,\lambda_d,r,w}(x) = C_{-1}e^{\frac{\lambda_d}{R_{-1}}x}$, $x < 0$, with $C_1 R_1 = C_{-1}R_{-1}$. To solve for the constants we need to proceed in a similar manner as above:

$$\begin{aligned}
\int f^{1}_{\lambda_b,\lambda_d,r,w}(x)dx + \int f^{-1}_{\lambda_b,\lambda_d,r,w}(x)dx &= 1 \\
\frac{C_1 R_1}{\lambda_d} + \frac{C_{-1}R_{-1}}{\lambda_d}(1 - e^{-\frac{\lambda_d}{(1-w).1 - wQ(\lambda_b,\lambda_d,r,w).}r}) &= 1 \\
C = \frac{\lambda_d}{2 - e^{-\frac{\lambda}{(1-w).1 - wQ(\lambda_b,\lambda_d,r,w).}r})}
\end{aligned}$$

For simplification of notation, we introduce auxiliary notation, $\lambda_1 = \frac{\lambda_d}{(1-w)+w.Q(\lambda_b,\lambda_d,r,w)}$ and $\lambda_2 = \frac{\lambda_d}{(1-w)-w.Q(\lambda_b,\lambda_d,r,w)}$. Hence, the density functions are denoted as follows, $f^{1}_{\lambda_b,\lambda_d,r,w}(x) = \frac{\lambda_1}{2-e^{-\lambda_2 r}}e^{-\lambda_1 x}$, $x > 0$ and $f^{-1}_{\lambda_b,\lambda_d,r,w}(x) = \frac{\lambda_2}{2-e^{-\lambda_2 r}}e^{\lambda_2 x}$, $x < 0$. Also, we can deduce that the marginal density $f_{\lambda_b,\lambda_d,r,w}(x) = f^{1}_{\lambda_b,\lambda_d,r,w}(x)$, $x > 0$ and $f_{\lambda_b,\lambda_d,r,w}(x) = f^{-1}_{\lambda_b,\lambda_d,r,w}(x)$, $x < 0$. Using the density computed above we can calculate $M(Q = 1) = \frac{1}{2-e^{-\lambda_2 r}}$ and $M(Q = -1) = \frac{1-e^{-\lambda_2 r}}{2-e^{-\lambda_2 r}}$. Also, the average quality needs to be consistent with the average quality computed using the distributions derived above. This is formally stated as

$$\bar{Q}(\lambda_b, \lambda_d, r, w) = \frac{e^{-\frac{\lambda_d r}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}}}{2 - e^{-\frac{\lambda_d r}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}}} = \frac{e^{-\lambda_2 r}}{2 - e^{\lambda_2 r}} \tag{1}$$

Next, we compute the total population mass by equating rate of births to the rate of deaths. The rate of deaths is comprised of two terms, the first term is the rate of natural deaths occurring due to Poisson shocks and the next term is the rate of deaths due to hitting the death boundary, $f_{\lambda_b,\lambda_d,r,w}(-r).(1 - w - w.\bar{Q}(\lambda_b, \lambda_d, r, w))$ corresponds to the density of the individuals hitting the death boundary per unit time. Hence, the rate of deaths is $\lambda_d.Pop(\lambda_b, \lambda_d, r, w) + f_{\lambda_b,\lambda_d,r,w}(-r).(1 - w - w.\bar{Q}(\lambda_b, \lambda_d, r, w)).Pop(\lambda_b, \lambda_d, r, w),$. Equating rate of births to rate of deaths we get the following.

$$\lambda_b = (\lambda_d + f_{\lambda_b,\lambda_d,r,w}(-r).(1 - w - w.\bar{Q}(\lambda_b, \lambda_d, r, w))Pop(\lambda_b, \lambda_d, r, w)$$
$$\lambda_b = (\lambda_d + \lambda_d.\frac{e^{-\lambda_2 r}}{2 - e^{-\lambda_2 r}}).Pop(\lambda_b, \lambda_d, r, w)$$

$$Pop(\lambda_b, \lambda_d, r, w) = \frac{\lambda_b}{\lambda_d(1 + \bar{Q}(\lambda_b, \lambda_d, r, w))} \tag{2}$$

Now that we have both the normalized density and the total population's expressions, we can arrive at the expression of the population density $p_{\lambda_b,\lambda_d,r,w}(x)$ which is just a product of the two, formally given as follows.

$$p_{\lambda_b,\lambda_d,r,w}(x) = \begin{cases} Pop(\lambda_b, \lambda_d, r, w).\frac{\lambda_1}{2-e^{-\lambda_2 r}}e^{-\lambda_1 x}, & \text{if } x > 0 \\ Pop(\lambda_b, \lambda_d, r, w).\frac{\lambda_2}{2-e^{-\lambda_2 r}}e^{\lambda_2 x}, & \text{if } x<0 \end{cases}$$

If we can show that there is a unique average quality, $\bar{Q}(\lambda_b, \lambda_d, r, w)$ satisfying (1) then both the total population mass (2) and the population density (3) are uniquely determined. We know that $Q \in \{-1, 1\}$ hence, $\bar{Q}(\lambda_b, \lambda_d, r, w) \in [-1, 1]$. To solve for $\bar{Q}(\lambda_b, \lambda_d, r, w)$, we need to solve $z = g(z)$, where $g(z) = \frac{e^{-\frac{\lambda_d r}{(1-w)-wz.}r}}{2-e^{-\frac{\lambda_d r}{1-w-w.z}r}}$ and $z \in [-1, 1]$. We will first show that there exists a solution in the set, $[-1, 1]$. Let $z_1 = -1$ and $z_2 = \min\{1, \frac{1-w}{w}\}$. If $w < \frac{1}{2}$ then, $z_2 = 1$ else $z_2 = \frac{1-w}{w}$. $g(z_1) = \frac{e^{-\frac{\lambda_d r}{1-w}}}{2-e^{-\frac{\lambda_d r}{1-w}}}$ and $g(z_1) > z_1$. If $w < \frac{1}{2}$ then $g(z_2) = \frac{e^{-\frac{\lambda_d r}{1-2w}}}{2-e^{-\frac{\lambda_d r}{1-2w}}}$ which is less than or equal to $z_2 = 1$, i.e. $g(z_2) \leq z_2$. Based on this and since the function z and $g(z)$ are continuous in the range $[-1, \frac{1-w}{w})$, there has to be a point in the interval $[-1, 1] \subset [-1, \frac{1-w}{w})$ where $g(z) = z$. Also, $g(z)$ is decreasing in the range $[-1, \frac{1-w}{w})$, this can be seen from the expression for $g^{'}(z) = -\frac{\lambda_d r w}{(1-w-wz)^2(2e^{\frac{\lambda_d r}{1-w-wz}}-1)^2}2e^{\frac{\lambda_d r}{1-w-wz}}$ and z is strictly increasing function. Therefore, $g(z) - z$ is a strictly decreasing function in $[-1, \frac{1-w}{w})$, which implies that the root is unique. When $w = \frac{1}{2}$, $z_2 = 1$ we can see that $g(z_1) > z_1$ holds, but $g(z)$ is not continuous at z_2. This is not a problem as we know that the function is continuous everywhere from $[-1, z_2)$ and $\lim^{'}_{z\to z_2} g(z) = 0$, where $\lim^{'}_{z\to z_2} g(z)$ corresponds to the left hand limit, hence $\lim^{'}_{z\to z_2} g(z) < z_2$. Hence, the same argument as above can be applied. In the case when $w > \frac{1}{2}$ then we will show that there exists a unique solution for $g(z) = z$ in the range $[-1, 1]$. We know that $g(z_1) = \frac{e^{-\frac{\lambda_d r}{1-w}}}{2-e^{-\frac{\lambda_d r}{1-w}}}$, but since $w > \frac{1}{2}$ we need to be careful about the case when $w = 1$. For now we can assume that $\frac{1}{2} < w < 1$. Hence, we know that $g(z_1) > z_1$. Here $z_2 = \frac{1-w}{w}$ and $g(z)$ will not be continuous at z_2. But we can show that $\lim'_{z\to z_2} g(z) = 0$, where $\lim'_{z\to z_2} g(z)$ corresponds to the left hand limit, and $\lim'_{z\to z_2} g(z) < z_2$. Hence, from the decreasing nature of $g(z) - z$ we know that there is a unique solution in the range $[-1, \frac{1-w}{w})$. Since $1 > w > \frac{1}{2}$ then $[-1, \frac{1-w}{w}) \subset [-1, 1]$ we need to show that there is no solution in the range $(\frac{1-w}{w}, 1]$. In the range $(\frac{1-w}{w}, 1]$ the function $g(z)$ is not necessarily continuous. There exists a discontinuity if $2e^{\frac{\lambda_d r}{1-w-wz}} - 1 = 0$ and $z \in (\frac{1-w}{w}, 1]$. Let's assume that there is a discontinuity. In that case, the function $g(z)$ will decrease values from -1 to $-\infty$, then to the right of the discontinuity at $2e^{\frac{\lambda_d r}{1-w-wz}} - 1 = 0$ the function decreases from ∞ to $\frac{1}{2e^{\frac{\lambda_d r}{1-2w}} - 1}$. Since $w > \frac{1}{2}$ and $2e^{\frac{\lambda_d r}{1-w-wz}} - 1 = 0$ for some $z \in (\frac{1-w}{w}, 1]$ $1 > 2e^{\frac{\lambda_d r}{1-2w}} - 1 > 0$ we can say that $\frac{1}{2e^{\frac{\lambda_d r}{1-2w}} - 1} > 1$. Hence, there is no point in the range in $[-1, 1]$ which intersects with this function. In the case, when there is no discontinuity it is straightforward to show that there is no solution of $g(z) = z$ as the function $g(z)$ will only take negative values less than -1. Also, when $w = 1$ the individuals welfare is fixed to zero all the time, hence there is a symmetry in the proportion of good and bad quality individuals, which leads to a unique solution $\bar{Q}(\lambda_b, \lambda_d, r, w) = 0$.

Lemma 1. Good and bad quality individuals attain positive and negative welfare values respectively.

Proof: The proof of theorem 1, already contains the proof for this lemma as we show that R_1 and R_{-1} attain positive and negative welfare values respectively.

Lemma 2. The average quality $\bar{Q}(\lambda_b, \lambda_d, r, w)$ and the average welfare $\bar{X}(\lambda_b, \lambda_d, r, w)$ of an individual a). Decrease as the level of collectivism, w is increased., b). Decrease as the rate of natural deaths, λ_d increases., c). Decrease as the the death boundary,$-r$ decreases.

Proof: We already know that the solution for $\bar{Q}(\lambda_b, \lambda_d, r, w)$ requires solving a transcendental equation (1), which means that we do not have a closed form analytical expression for it. It can be shown that the expression for $\bar{X}(\lambda_b, \lambda_d, r, w)$ expressed in terms of the $\bar{Q}(\lambda_b, \lambda_d, r, w)$ is $(r + \frac{1}{\lambda_d}).\bar{Q}(\lambda_b, \lambda_d, r, w)$. From Theorem 1, we know that for every set of parameters there does exist a solution $\bar{Q}(\lambda_b, \lambda_d, r, w)$. For part a), as the level of collectivism is increased let us assume that the average quality $\bar{Q}(\lambda_b, \lambda_d, r, w)$ increases. However, if there is an increase in both the collectivism and the average quality, the expression $g(\bar{Q}(\lambda_b, \lambda_d, r, w)) = \frac{e^{-\frac{\lambda_d r}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}}}{2-e^{-\frac{\lambda_d r}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}}}$ decreases which contradicts the increase in $\bar{Q}(\lambda_b, \lambda_d, r, w)$. Hence, $\bar{Q}(\lambda_b, \lambda_d, r, w)$ has to decrease with an increase in collectivism. And from the expression of $\bar{X}(\lambda_b, \lambda_d, r, w)$ expressed in terms of $\bar{Q}(\lambda_b, \lambda_d, r, w)$ it is straightforward that the average welfare also decreases with an increase in the level of collectivism. For part b), again as the rate of natural deaths increases assume that $\bar{Q}(\lambda_b, \lambda_d, r, w)$ increases. However the decrease in $\frac{e^{-\frac{\lambda_d r}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}}}{2-e^{-\frac{\lambda_d r}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}}}$ will contradict the assumption. With an increase in λ_d the first term in the expression of $\bar{X}(\lambda_b, \lambda_d, r, w)$ which inversely related to λ_d has to decrease, this combined with the decrease in $\bar{Q}(\lambda_b, \lambda_d, r, w)$ leads to a decrease in the average welfare. For part c), we arrive at the expression of the derivative of average quality $\bar{Q}(\lambda_b, \lambda_d, r, w)$ w.r.t. r, $-\frac{\lambda_d(d)(d+1)(1-w-wd)}{(1-w-wd)^2+\lambda_d r w(d+1)}$ which is negative. Hence, we know that the average quality indeed decreases with an increase in r. For average welfare we give an intuitive explanation first, increasing r decreases the average quality as a result of which the growth of a good quality individual slows down and the decay of a bad quality individual becomes faster. As a result the average welfare levels attained by a good and bad quality individual are lower. Moreover, increase in r increases the proportion of the bad quality individuals which further has a negative effect on the average welfare. To prove this formally we will show that the average welfare of both good and bad quality individuals decreases and the proportion of the bad quality individuals increases. Since the average welfare value of a bad quality individual is always lower than that of a good quality individual this is sufficient to show the result. The average welfare of good quality individuals is given as $\frac{1}{\lambda_1} = \frac{1-w+w\bar{Q}(\lambda_b,\lambda_d,r,w)}{\lambda_d}$. This can be derived as follows, the distribution of the welfare conditional on the fact that individuals are of good quality $f_{\lambda_b,\lambda_d,r,w}(x|Q=+1)$ can be shown to be an exponential distribution with parameter λ_1 exactly on the same lines as we derived the joint densities $f^1_{\lambda_b,\lambda_d,r,w}(x)$ in Theorem 1. Since $\bar{Q}(\lambda_b,\lambda_d,r,w)$ decreases as a function of r, the average welfare of a good quality individual also decreases as a function of r. Similarly we need to arrive at the distribution $f_{\lambda_b,\lambda_d,r,w}(x|Q=-1)$,which turns out to be $f_{\lambda_b,\lambda_d,r,w}(x|Q=-1) = \frac{\lambda_2}{1-e^{-\lambda_2 r}}e^{\lambda_2 x}$, $x<0$. The average welfare value of bad quality individual can be arrived at using this distribution and it turns out to be, $-\frac{1}{\lambda_2}+\frac{re^{-\lambda_2 r}}{1-e^{-\lambda_2 r}}$. As r is increased, $\bar{Q}(\lambda_b,\lambda_d,r,w)$ decreases and thus λ_2 decreases as well. The partial derivative of average welfare of bad quality individual $-\frac{1}{\lambda_2}+\frac{re^{-\lambda_2 r}}{1-e^{-\lambda_2 r}}$ w.r.t. r is given as $\frac{e^{\lambda_2 r}-\lambda_2 r e^{\lambda_2 r}-1}{(e^{\lambda_2 r}-1)^2}$ and this expression turns out to be negative for $(\lambda_2, r)\in\mathbb{R}^2_+$. Also, it can be shown that the partial derivative of $-\frac{1}{\lambda_2}+\frac{re^{-\lambda_2 r}}{1-e^{-\lambda_2 r}}$ w.r.t λ_2 is given as $(\frac{1}{\lambda_2})^2-\frac{r^2e^{\lambda_2 r}}{(e^{\lambda_2 r}-1)^2}$ and this expression turns out to be positive. Hence, from the sign of these partial derivatives we can easily see the result.

Theorem 2. a) Total population $Pop(\lambda_b,\lambda_d,r,w)$ increases as the rate of birth λ_b increases. b) $Pop(\lambda_b,\lambda_d,r,w)$ increases as the level of collectivism w increases. c) $Pop(\lambda_b,\lambda_d,r,w)$ increases as the death boundary$-r$ decreases. d) If $w<\frac{1}{2}$ then $Pop(\lambda_b,\lambda_d,r,w)$ increases as the rate of natural deaths λ_d decreases.

Proof: In order to compute the total population in the steady state, we need to have the rate of birth equals the rate of death which is formally stated as follows,

$$(\lambda_d + f_{\lambda_b,\lambda_d,r,w}(-r).(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w))).Pop(\lambda_b,\lambda_d,r,w)=\lambda_b$$

$$Pop(\lambda_b,\lambda_d,r,w)=\frac{\lambda_b}{\lambda_d.(1+\bar{Q}(\lambda_b,\lambda_d,r,w))}$$

For part a), $\bar{Q}(\lambda_b,\lambda_d,r,w)$ does not depend on the rate of births and it is clear that the result holds since the population is directly proportional to λ_b. For part b) as well it can be seen that the only term in the expression which depends on w is $\bar{Q}(\lambda_b,\lambda_d,r,w)$ which will decrease as w is increased (Lemma 2). Therefore, it is clear that the population has to increase with level of collectivism. For part c), again we can see that the only term in the expression which depends on the death boundary $-r$ is $\bar{Q}(\lambda_b,\lambda_d,r,w)$. We know that as the death boundary decreases $\bar{Q}(\lambda_b,\lambda_d,r,w)$ decreases as well (Lemma 2), thereby leading to an increase in the population. In part d), as the rate at which natural deaths occur decreases, the rate of deaths due to achieving poor welfare levels or hitting the death boundary can increase. However, if the level of dependence on the society is low then the decrease in the rate of natural deaths dominates, as a result the total population increases such that the mass of deaths equals mass of birth. We now show this formally. Let us take the derivative of the term in the denominator w.r.t λ_d,

$$(1+\bar{Q}(\lambda_b,\lambda_d,r,w))+\frac{d\bar{Q}(\lambda_b,\lambda_d,r,w)}{d\lambda_d}$$

$$(1+\bar{Q}(\lambda_b,\lambda_d,r,w))(\frac{(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w))^2+\lambda_d rw\bar{Q}(\lambda_b,\lambda_d,r,w)-\lambda_d r\bar{Q}(\lambda_b,\lambda_d,r,w)(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w))}{(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w))^2+\lambda_d rw\bar{Q}(\lambda_b,\lambda_d,r,w)}$$

$$(1+\bar{Q}(\lambda_b,\lambda_d,r,w))(\frac{(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w))(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)-\lambda_d r\bar{Q}(\lambda_b,\lambda_d,r,w))+\lambda_d rw\bar{Q}(\lambda_b,\lambda_d,r,w)}{(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w))^2+\lambda_d rw\bar{Q}(\lambda_b,\lambda_d,r,w)})$$

If we can show that $(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)-\lambda_d r\bar{Q}(\lambda_b,\lambda_d,r,w))>0$ then the above expression will be positive. We know from lemma 2 that $\bar{Q}(\lambda_b,\lambda_d,r,0)\geq\bar{Q}(\lambda_b,\lambda_d,r,w),\ \forall w\in[0,1]$. This leads to $\bar{Q}(\lambda_b,\lambda_d,r,0)<\frac{1-w}{w+\lambda_d r}$ which is a sufficient for the above derivative to be positive. It can be checked that this condition is satisfied if $w<\frac{1}{2}$.

Theorem 3: a) Cumulative welfare $CF(\lambda_b,\lambda_d,r,w)$ decreases as the rate of birth λ_b decreases. b) $CF(\lambda_b,\lambda_d,r,w)$ decreases as the rate of natural deaths λ_d increases. c) If $\lambda_d r\leq\epsilon<\frac{1}{2}$ & $w<\frac{1}{2}-\epsilon$ with $\epsilon>0$, then $CF(\lambda_b,\lambda_d,r,w)$ decreases as the death boundary $-r$ decreases. d) $CF(\lambda_b,\lambda_d,r,w)$ decreases as the level of collectivism w increases.

Proof: For part a), we know that $CF(\lambda_b,\lambda_d,r,w)=\bar{X}(\lambda_b,\lambda_d,r,w)Pop(\lambda_b,\lambda_d,r,w)$. Also, since the average welfare of an individual is independent of λ_b we only need to consider the effect on total population which we already know from Theorem 2. For part b), let us simplify the expression of cumulative welfare, $CF(\lambda_b,\lambda_d,r,w)=(r+\frac{1}{\lambda_d}).\frac{\lambda_b}{\lambda_d}.\frac{\bar{Q}(\lambda_b,\lambda_d,r,w)}{(1+\bar{Q}(\lambda_b,\lambda_d,r,w))}$. From this expression we can see that as λ_d increases the term $(r+\frac{1}{\lambda_d}).\frac{\lambda_b}{\lambda_d}$ will definitely decrease. In fact the other term will also decrease, as can be seen from the derivative of the second term w.r.t. λ_d , $\frac{1}{(1+\bar{Q}(\lambda_b,\lambda_d,r,w))^2}\frac{d\bar{Q}(\lambda_b,\lambda_d,r,w)}{d\lambda_d}$ and this combined with Lemma 2. For part d), we can see that only $\frac{\bar{Q}(\lambda_b,\lambda_d,r,w)}{(1+\bar{Q}(\lambda_b,\lambda_d,r,w))}$ depends on the weight w and its derivative w.r.t. w is $\frac{1}{(1+\bar{Q}(\lambda_b,\lambda_d,r,w))^2}\frac{d\bar{Q}(\lambda_b,\lambda_d,r,w)}{dw}$. This expression of the derivative and Lemma 2, lead us to the result. For part c), as the death boundary decreases, the total population in the society increases whereas the average welfare of an individual decreases, leading to opposing effects. Therefore, if the $\lambda_d r$ is sufficiently low then the proportion of the population with bad quality is sufficiently low as well. Also, if the level of collectivism, w is low then then the rate at which the welfare of bad quality individuals decays with time is high, hence the effect of decreasing the death boundary on the average welfare is high. Under these conditions the decrease in average welfare dominates the increase in population. We next show this formally. The derivative of cumulative welfare w.r.t. r is given as,

$$\Big(\frac{\bar{Q}(\lambda_b,\lambda_d,r,w)}{\bar{Q}(\lambda_b,\lambda_d,r,w)+1}\Big).(\frac{(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)^2-(\lambda_d r+1)(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)))}{(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)^2+\lambda_d rwd(d+1)})$$

$$\Big(\frac{\bar{Q}(\lambda_b,\lambda_d,r,w)}{\bar{Q}(\lambda_b,\lambda_d,r,w)+1}\Big).(\frac{(w.(\bar{Q}(\lambda_b,\lambda_d,r,w)).(-(1-w.(1+\bar{Q}(\lambda_b,\lambda_d,r,w))+\lambda_d r+\lambda_d r\bar{Q}(\lambda_b,\lambda_d,r,w))-\lambda_d r}{(1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)^2+\lambda_d rwd(d+1)})$$

If $-(1 - w.(1 + \bar{Q}(\lambda_b, \lambda_d, r, w)) + \lambda_d r + \lambda_d r\bar{Q}(\lambda_b, \lambda_d, r, w) < 0$ then the above derivative is negative. Note $\bar{Q}(\lambda_b, \lambda_d, r, 0) < \frac{1-w-\lambda_d r}{w+\lambda_d r}$ is sufficient for this condition to hold and it leads to the following condition, $w < \frac{1}{2} - \epsilon$ and $\lambda_d r \leq \epsilon < \frac{1}{2}$ where $\epsilon > 0$. This proves part c.

Theorem 4. a) Average life time $\bar{T}(\lambda_b, \lambda_d, r, w)$ decreases with an increase in rate of natural deaths λ_d. b) If $\lambda_d r > \theta^* = \ln(1 + \frac{\sqrt{2}}{2})$[1], then $\bar{T}(\lambda_b, \lambda_d, r, w)$ increases with an increase in level of collectivism w else, it first decreases and then increases with an increase in level of collectivism w. c), If $\lambda_d r > \theta^*$, then $\bar{T}(\lambda_b, \lambda_d, r, w)$ increases with a decrease in death boundary $-r$ else, it first decreases and then increases with a decrease in death boundary $-r$.

Proof: The expression for the average life-time of an individual $\bar{T}(\lambda_b, \lambda_d, r, w)$ involves the computation of the average life-time of good quality individuals and bad quality individuals separately and then combining the two using the conditional probabilities. Hence, $\bar{T}(\lambda_b, \lambda_d, r, w) = \frac{1}{\lambda_d} + (\frac{1}{\lambda_d})\frac{\bar{Q}(\lambda_b,\lambda_d,r,w)-\bar{Q}(\lambda_b,\lambda_d,r,w)^2}{\bar{Q}(\lambda_b,\lambda_d,r,w)+1}$. The derivative of $\bar{T}(\lambda_b, \lambda_d, r, w)$ w.r.t w can be expressed as $\frac{1}{\lambda_d}\frac{\bar{Q}(\lambda_b,\lambda_d,r,w)^2+2\bar{Q}(\lambda_b,\lambda_d,r,w)-1}{(\bar{Q}(\lambda_b,\lambda_d,r,w)+1)^2}\frac{d\bar{Q}(\lambda_b,\lambda_d,r,w)}{dw}$. If $\bar{Q}(\lambda_b, \lambda_d, r, 0) < \sqrt{2}-1$ then the above derivative is positive. This leads to the condition $\lambda_d r > \ln(1 + \frac{\sqrt{2}}{2})$. However, if $\lambda_d r < \ln(1 + \frac{\sqrt{2}}{2})$ then $\bar{Q}(\lambda_b, \lambda_d, r, 0) > \sqrt{2}-1$ and as a result the derivative is negative. However, $\bar{Q}(\lambda_b, \lambda_d, r, w)$ will decrease with increase in w and it can be observed that at $w = 1$, $\bar{Q}(\lambda_b, \lambda_d, r, w)$ will be zero, this is due to the fact that the individuals completely depend on the society and the rate of growth is zero for all individuals. Hence, for some $w = w^*$ the $\bar{Q}(\lambda_b, \lambda_d, r, w^*) = \sqrt{2}-1$ where the life-time will take the minimum value. Therefore, we know that in the region $w > w^*$, the life-time will increase. This explains part b). For part c), a similar explanation can be given. The expression for the derivative changes to $\frac{1}{\lambda_d}\frac{\bar{Q}(\lambda_b,\lambda_d,r,w)^2+2\bar{Q}(\lambda_b,\lambda_d,r,w)-1}{(Q(\lambda_b,\lambda_d,r,w)+1)^2}\frac{d\bar{Q}(\lambda_b,\lambda_d,r,w)}{dr}$ and the rest of the explanation follows from above and Lemma 2. For part a), we will first show that the average life-time of both a good and bad quality individual decrease. Then, we will show that the proportion of the bad quality individuals increase. Since the average life-time of a bad quality individual is always lesser than that of a good quality individual, this will lead to a decrease in the average life-time unconditional on the quality of the individual. First of all the average life-time of a good quality individual is $\frac{1}{\lambda_d}$ and it decreases with λ_d. Next, the average life-time of an individual with bad quality is arrived at by computing the expectation of $\min\{T^{'}, T_2(\lambda_b, \lambda_d, r, w) = \frac{r}{1-w(1+\bar{Q}(\lambda_b,\lambda_d,r,w))}, \}$ where $T^{'}$ is an exponential random variable with mean $\frac{1}{\lambda_d}$. The life-time of a bad quality individual is $\frac{1}{\lambda_d}.(1-e^{-\frac{\lambda_d r}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}})$, the derivative of this expression is $-\frac{1}{\lambda_d^2}.(1 - e^{-\frac{\lambda_d r}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}} - \frac{\lambda_d r}{1-w-w\bar{Q}(\lambda_b,\lambda_d,r,w)}e^{-\frac{\lambda_d r}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}}) + \frac{r.w}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}\frac{d\bar{Q}(\lambda_b,\lambda_d,r,w)}{d\lambda_d}$. The term $(1-e^{-\frac{\lambda_d r}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}} - \frac{\lambda_d r}{1-w-w\bar{Q}(\lambda_b,\lambda_d,r,w)}e^{-\frac{\lambda_d r}{1-w-w.\bar{Q}(\lambda_b,\lambda_d,r,w)}})$ has to be positive since $(x+1)e^{-x} < 1$. Hence, we can see that the derivative is negative which implies the result.

Theorem 5. The average inequality $Var_X(\lambda_b, \lambda_d, r, w)$ is always more in an individualistic society $w = 0$ as compared to a collectivistic society $w = 1$. Also if the person only dies a natural death, i.e. $r \to \infty$, then a) $\lim_{r\to\infty} Var_X(\lambda_b, \lambda_d, r, w)$ decreases with an increase in level of collectivism w and b) $\lim_{r\to\infty} Var_X(\lambda_b, \lambda_d, r, w)$ decreases with an increase in rate of natural deaths λ_d.

[1] θ^* is a fixed constant which in general will depend on $P(Q = 1)$, and when $P(Q = 1) = \frac{1}{2}$ it is $\ln(1 + \frac{\sqrt{2}}{2})$.

Proof: $Var_X(\lambda_b, \lambda_d, r, w = 1) = 0$ since all the individuals have the same welfare value of zero. So, we need to show that $Var_X(\lambda_b, \lambda_d, r, w = 0) > 0$. The expression for variance is,

$$Var_X(\lambda_b, \lambda_d, r, w = 0) = (\frac{1}{\lambda_d})^2 \frac{(8e^{2\lambda_d r} + e^{\lambda_d r}(-2(\lambda_d r)^2 + 4\lambda_d r - 8) - 3\lambda_d r + (\lambda_d r)^2 + 1)}{(2e^{\lambda_d r} - 1)^2}$$

It can be shown that the expression in the numerator of the above expression is indeed positive. To do so we show that at any point $(\lambda_d, r) \in \mathbb{R}_+^2$ the partial derivative w.r.t to either λ_d or r is positive and also that $Var_X(\lambda_b, \lambda_d = 0, r = 0, w = 0) > 0$ which helps us establish the result.

For part a), the case when an individual only dies a natural death there is a symmetry in the proportion of individuals with good and bad quality. Hence, the average quality of an individual is zero. Therefore, the rate of decay for an individual with bad quality is $1 - w$ and the same is the rate of growth for an individual with good quality. Hence, increasing w slows the rate of decay and growth, thereby allowing individuals to neither take too low or too high welfare values, which leads to a lower average disparity. Formally if $r \to \infty$, $\bar{Q}(\lambda_b, \lambda_d, r, w) = 0$, this leads to the density distribution given as, $f^1_{\lambda_b,\lambda_d,r,w}(x) = \frac{\lambda_d}{1-w} e^{-\frac{\lambda_d}{1-w}x}$ and $f^{-1}_{\lambda_b,\lambda_d,r,w}(x) = \frac{\lambda_d}{1-w} e^{\frac{\lambda_d}{1-w}x}$. This leads to the expression of the variance given as, $(\frac{1-w}{\lambda_d})^2$ and therefore, part a) and b) follow directly from this.