ONLINE APPENDIX: STRATETIC INFORMATION DISSEMINATION AND NETWORK FORMATION

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APPENDIX A

(1) Proof of the existence of NE

It should be noted that the strategy space for each agent in the IDG is compact and convex. Meanwhile, an agent's utility is quasi-concave over its link formation strategy. Hence, it has been shown in [1] that pure Nash equilibrium always exists in such a game. ■

(2) Proof of Corollary 1

This can be proved using the same idea as Proposition 3. We first prove the following claim.

Claim 1. Given a strict NE $\boldsymbol{g}^*$ and when $k_{ij} = k, \forall i, j \in N$, if $g_{ij}^* = 1$ for some $i, j \in N$, then $\overline{g}_{jj'}^* = 0$ for any $j' \neq i$ and $j' \neq j$.

Proof of Claim 1: Suppose, in contrast, $g_{ij}^* = 1$ and $\overline{g}_{jj'}^* = 1$ for some $j' \neq i$ and $j' \neq j$. Then by deleting its link with j and forming a new link with j', i receives the same payoff as what it receives in $\boldsymbol{g}^*$, which contradicts the fact that $\boldsymbol{g}^*$ is an (strict) equilibrium and hence this lemma follows. ■

Then we are able to show that for each non-singleton component always has a star topology in a strict NE.

Without loss of generality, we select two agents $i, j \in C$ where C is a component in $cl(\boldsymbol{g}^*)$, such that $g_{ij}^* = 1$, then according to Claim 1, we have that $\overline{g}_{jj'}^* = 0$ for any $j' \in C$ and $j' \notin \{i, j\}$. According to Proposition 1, we should also have $g_{ji}^* = 0$ since otherwise agent j can strictly increase its utility by removing the link it forms to agent i.

Now suppose that $g_{j'i}^* = 1$ for some $j' \in C$ and $j' \notin \{i, j\}$. Then it is obvious that agent j' can switch its link from agent i to agent j without decreasing its utility, which gives a contradiction. Therefore, we can conclude that $g_{ij}^* = 1, \ \forall j \in C \text{ and } j \neq i$. Meanwhile, $g_{jj'}^* = 0, \ \forall j, j' \in C \text{ and } j, j' \neq i$. In other words, C has a star topology where agent i stays in the center and forms links with all other agents who stay in the periphery, while all the other agents do not form links mutually. This corollary thus follows. ■

(3) Proof of Theorem 2

(i) When $f(x) < \underline{k}$, suppose there is an equilibrium $\boldsymbol{g}^*$ which contains a non-singleton component C. Let i and j be two agents in C such that $g^*_{ij} = 1$, it is obvious that agent i can strictly increase its utility by setting $g_{ij} = 0$. Hence there is a contradiction and this statement follows.

(ii) When $f(x) \in (\underline{k}, \overline{k})$, consider a component C (which is minimal according to Proposition 1) and one of its periphery agent i such that $\overline{g}^*_{ji} = 1, \exists j$ and $\overline{g}^*_{j'i} = 0, \forall j' \neq j$.

Suppose $g^*_{ij} = 1$: If $k_{ij} = \overline{k}$ and $|C| > 2$, agent i can always switch its link to some other agent $j' \in C$ without decreasing its utility, If $k_{ij} = \overline{k}$ and $|C| = 2$, agent i can always increase its utility by switching its link to some other agent $j' \notin C$. In both cases contradict the fact that $\boldsymbol{g}^*$ is a strict NE. Hence, we have $k_{ij} = \underline{k}$. Hence, we have $g^*_{i'j} = 0, \forall i' \in C / \{i, j\}$ (otherwise i' can switch its link from j to i without decreasing its utility). Since i is a periphery agent, we thus have $g^*_{ji'} = 1, \exists i' \in C / \{i, j\}$. If $|C| = 3$, then $k_{ji'} = \underline{k}$ and agent i can switch its link from j to i' without decreasing its utility. Therefore, we have $|C| > 3$ and $\overline{g}^*_{i'i''} = 1, \exists i'' \in C / \{i, j, i'\}$. If $k_{ji'} = \overline{k}$, agent j can switch its link from i' to i'' without decreasing its utility, whereas if $k_{ji'} = \underline{k}$, agent i can switch its link from j to i' without decreasing its utility. Both cases contradict the fact that $\boldsymbol{g}^*$ is a strict NE. It can be thus concluded that $g^*_{ij} = 1$ cannot hold in $\boldsymbol{g}^*$ and we have $g^*_{ji} = 1$. As a result, j should belong to the same group as i with $k_{ij} = \underline{k}$.

Now consider any other agent $j' \in C / \{i, j\}$. If $g^*_{j'j} = 1$, then j' can switch its link from j to i without decreasing its utility, which leads to a contradiction. Therefore, we have $g^*_{jj'} = 1, \forall j' \in C / \{j\}$ and the component forms a star topology. Also, if there is an agent $j'' \in C$ who is not from the same group as j, then j can always increase its utility by removing the link to j'' since $f(x) < \overline{k}$. Hence, this statement follows.

(iii) When $f(x) > \overline{k}$, it is still true that agents from the same group belong to the same component. Also, the network should be connected with a unique component under $\boldsymbol{g}^*$. It is always true that we can find two agents i and i' from one group N_z such that $g^*_{ii'} = 1$. Using the same argument as in statement (ii), it is easy to show that $g^*_{ii''} = 1, \forall i'' \in N_z / \{i\}$. Now consider an agent $j \notin N_z$. We have $g^*_{ji'} = 0, \forall i' \in N_z$ (otherwise the condition of a strict NE is violated). Now consider a path $path_{i'j} = ((i', j_1), (j_1, j_2), ..., (j_m, j))$ with $j_1, ..., j_m, j \notin N_z$ and $i' \in N_z$. Obviously, we have $g^*_{i'j_1} = 1$. Then i' can switch its link from j_1 to j without decreasing its utility, which again violates the fact that $\boldsymbol{g}^*$ is a strict NE. It can be thus concluded that for each $j \notin N_z$, $g^*_{i'j} = 1, \exists i' \in N_z$ and $g^*_{jj'} = 0, \forall j' \notin N_z$. Hence, this statement follows. ■

(4) Proof of Proposition 3

This can be proven using the same idea as Proposition 1. ■

(5) Proof of Theorem 3

Let $k_{i_0} = \min_{i \in N} k_i$ and consider a periphery-sponsored star $\boldsymbol{g}$ with $g_{ji_0} = 1, \forall j \in N / \{i_0\}$ and $g_{jj'} = 0, \forall j, j' \in N / \{i_0\}$. It is obvious that $\boldsymbol{g}$ is both social optimal and an NE. Hence, this theorem is proven. ■

(7) Proof of Proposition 4

We consider an arbitrary NE $\boldsymbol{g}^*$. Let $k_{i_0} = \min_{i \in N} k_i$ and consider a component C_1 that contains i_0. Now consider another component C_2. If C_2 is a singleton component which contains a unique agent j, then j can always increase its utility by forming a link with i_0. If C_2 is a non-singleton component, then there is always an agent $j' \in C_2$ such that $g^*_{j'l} = 1, \exists l \in C_2$ with $\overline{g}^*_{ll'} = 0, \forall l' \in N / \{l\}$. In this case, j' can also increase its utility by switching its link from l to i_0. Therefore, it can be concluded that $\boldsymbol{g}^*$ forms a connected network. From Proposition 3, we know that the network formed by $\boldsymbol{g}^*$ is also minimal which contains $|N|-1$ links and hence, $U^\# / U(\boldsymbol{g}^*) \leq \max_{i,j \in N} k_i / k_j$. Since this conclusion applies to any NE $\boldsymbol{g}^*$, Proposition 4 thus follows. ■

(8) Proof of Corollary 2

This can be proved straightforwardly using Proposition 1 and 3. ■

APPENDIX B

(9) Proof of Proposition 5

(i) Consider an agent i with $\overline{g}_{ij}^* = 0, \forall j \in N / \{i\}$. Suppose $x_i^* > 0$, then agent i's utility is $f(0) - cx_i^*$. By setting $x_i = 0$, agent i receives a utility $f(0) > f(0) - cx_i^*$, which contradicts the fact that $\boldsymbol{s}^*$ is an equilibrium. Hence, we have $x_i^* = 0$ for any agent i with $\overline{g}_{ij}^* = 0, \forall j \in N / \{i\}$.

Now consider an agent j with $\overline{g}_{jj'}^* = 1, \exists j' \in N / \{j\}$. Suppose agent $x_j^* = 0$, agent j's utility is $f(0) - \sum_{j' \in N_j(\boldsymbol{g})} k_{jj'}$. Given $f'(0) > c$, there is always an value ε such that $f(\varepsilon \mid N_j(\overline{\boldsymbol{g}}) \mid) - c\varepsilon > f(0)$. Therefore, agent j's utility increases with $x_j = \varepsilon$, which contradicts the fact that $\boldsymbol{s}^*$ is an equilibrium. Hence we have $x_j^* > 0$ for any agent j with $\overline{g}_{jj'}^* = 1, \exists j' \in N / \{j\}$.

(ii) This can be proven using the same argument as in statement (i). ■

(10) Proof of Theorem 4

To prove Statement (i), it is sufficient to see that each agent i will connect to at least one other agent in any NE and thus have $x_i^* > 0$ when $\max_{ij} k_{ij} < c\overline{x}$, which is independent to the population size $|N|$.

By taking the first order derivative of the utility function (4) over x_i, we have that $| N_i(\overline{\boldsymbol{g}}^*) | f'(| N_i(\overline{\boldsymbol{g}}^*) | x_i^*) = c$ and thus $| N_i(\overline{\boldsymbol{g}}^*) | x_i^* \geq \overline{x}$. Also, for any two agents i, j within a same component, we have $x_i^* = x_j^*$. Therefore, for any component C, the total amount of information produced by agents within this component at equilibrium is $\sum_{i \in C} x_i^*$ which satisfies $f'(\sum_{i \in C} x_i^*) = c / |C|$ and $\sum_{i \in C} x_i^* > \overline{x}$. Suppose that there is a sufficiently large constant W such that for any N we have $\inf_{\boldsymbol{s}^* \in S_N^*} \{\sum_{i \in N} x_i^*\} < W$. Select $\boldsymbol{s}_N^* = \arg \inf_{\boldsymbol{s}^* \in S_N^*} \{\sum_{i \in N} x_i^*\}$. Due to the concavity of $f(\cdot)$, we have that $f'(\sum_{i \in C} x_i^*) = c / |C| \geq f'(W)$ for any component C under $\boldsymbol{s}_N^*$. Hence, $|C| \leq c / f'(W)$. Then we have $\sum_{i \in N} x_i^* \geq |N| f'(W) \overline{x} / c$. This shows that there is always a sufficiently large $|N|$ such that

$|N| f'(W)\overline{x} / c > W$ which contradicts the assumption that $\inf_{s^* \in S_N^*} \{\sum_{i \in N} x_i^*\} < W$ for any N. Therefore, we have a contradiction and Statement (ii) follows. ■

(11) Proof of the convergence under the best response dynamic (7)

Given the best response dynamic introduced by (7), the strategy profile $\boldsymbol{g}$ evolves following a Markovian process over a finite state space. Also, each strict equilibrium represents a steady state (i.e. an attractor in this Markov chain). Therefore, there are two possible scenarios after sufficient long time. In the first scenario, $\boldsymbol{g}$ converges to a strict equilibrium. In the second scenario, $\boldsymbol{g}$ enters a loop of states and never exits the loop again. With positive inertia probabilities $\{r_i\}_{i=1}^n$, it has been shown in [2] that the second scenario never takes place and hence $\boldsymbol{g}$ converges to a strict equilibrium with probability 1 in the long run. ■

[1] P. Reny, "On the existence of pure and mixed strategy Nash equilibria in Discontinuous games," *Econometrica*, vol. 67, no. 5, pp. 1029 – 1056, 1999.

[2] M. Jackson and D. Watts, "The evolution of social and economic networks," *Journal of Economic Theory*, vol. 106, no. 2, pp. 265 – 295, 2002.