Online Appendix for “Opportunistic Information Networks”

Ahmed M. Alaa, Member, IEEE, Kartik Ahuja, and Mihaela van der Schaar, Fellow, IEEE

APPENDIX A

PROOF OF THEOREM 1

From Nash’s Existence Theorem, we know that if we allow mixed strategies, then every game with a finite number of players in which each player can choose from finitely many pure strategies has at least one Nash equilibrium [28]. Assume that agent $i$ of players in which each player can choose from finitely many pure strategies has at least one Nash equilibrium forms a link with agent $i$, and $p_{ij}$ is the probability that agent $i$ forms a link with agent $j$, and $p_{ii} = 0, \forall i \in N$. The utility function of agent $1$ is given by

$$\Delta_1 = \sum_{j=1}^{2} w_j f(H(X_1 \cup X_j)) - \sum_{l=1}^{N-1} p_{ii}c,$$

where $\alpha_j$ is an element of the power set of $N/\{i\}$, and $w_j$ is the probability of the emergence of a network component comprising agents in the set $\{i \cup \alpha_j\}$ based on the mixed strategies. For instance, in a 2 agent network, the utility function of agent $1$ is given by

$$u_1(\Delta_1) = (p_{12}(1-p_{21}) + p_{21}(1-p_{12}) + p_{12}p_{21}) f(H(X_1, X_2))$$

$$+ (1-p_{12})(1-p_{21}) f(H(X_1)) - p_{12}c.$$ 

In this case, $w_1 = p_{12}(1-p_{21}) + p_{21}(1-p_{12}) + p_{12}p_{21}$ and $w_2 = (1-p_{12})(1-p_{21})$. Let the NE strategy profile be $\Delta^* = (\Delta^*_1, \Delta^*_2, ..., \Delta^*_N)$, where $\Delta^*_i = (p^*_1, p^*_2, ..., p^*_N)$. According to (3), the following condition on $\Delta^*$ needs to be satisfied

$$u_i(\Delta^*_i, \Delta^{*\alpha}_j) \geq u_i(\Delta_i, \Delta^{*\alpha}_j), \forall \Delta_i \in [0, 1]^N, \forall i \in N.$$  \hspace{1cm} (A.2)

Now we show that for any agent $i$, the NE strategy $\Delta^*_i$ needs to be a pure strategy for condition (A.2) to be satisfied. We focus on agent $i$ with a NE strategy $\Delta^*_i = (p^*_{i1}, p^*_{i2}, ..., p^*_{iN})$, where $p^*_{ij} \in [0, 1]$. Now assume we induce a perturbation $\epsilon$ to the mixed strategy of agent $i$ by modifying $p^*_{ik}$ to $p^*_{ik} + \epsilon$ for a certain $k$, where $\epsilon \in [-p^*_{ik}, 1 - p^*_{ik}]$. We call this modified strategy $\Delta^*_i(\epsilon)$. Note that we can write any $w_j$ in (A.1) in the form of $w_j = \tilde{w}_j p^*_ik + \bar{w}_j (1-p^*_ik)$. This results in a perturbed utility $u_i(\Delta^*_i(\epsilon))$ as follows

$$u_i(\Delta^*_i(\epsilon)) = \sum_{j=1}^{2N-1} (\tilde{w}_j (p^*_ik + \epsilon) f(H(X_1 \cup X_j))).$$

The authors are with the Department of Electrical Engineering, University of California Los Angeles (UCLA), Los Angeles, CA, 90095, USA (e-mail: ahmedmalaa@ucla.edu, ahujak@ucla.edu, mihaela@ee.ucla.edu). This work was funded by the Office of Naval Research (ONR).

July 31, 2015
which can be rearranged as

\[ u_i(\Delta^*_i(\epsilon)) = u_i(\Delta^*_i) + \epsilon \left( \sum_{j=1}^{2^{N-1}-1} (\tilde{w}_j - \hat{w}_j) \beta (H(X_i \cup X_{\alpha_j})) - c \right). \]

Let \( \delta = \sum_{j=1}^{2^{N-1}-1}(\tilde{w}_j - \hat{w}_j)\beta (H(X_i \cup X_{\alpha_j})) - c \). It can be easily shown that \( \frac{\partial u_i(\Delta^*_i(\epsilon))}{\partial \epsilon} > 0 \) if \( \delta > 0 \), and \( \frac{\partial u_i(\Delta^*_i(\epsilon))}{\partial \epsilon} < 0 \) otherwise. Thus, if \( \delta > 0 \), agent \( i \) can always increase its utility by increasing \( \epsilon \) and setting \( \epsilon = 1 - p^*_i \) (and thus playing a pure strategy with \( p^*_i = 1 \)), and if \( \delta < 0 \), agent \( i \) can always increase its utility by setting \( \epsilon = -p^*_i \) (and thus playing a pure strategy with \( p^*_i = 0 \)), which contradicts with \( \Delta^*_i \) being a NE strategy. Thus, for all \( k \in \mathcal{N}/\{i\} \), agent \( i \) needs to select a pure strategy \( p^*_k \in \{0,1\} \) for \( \Delta^*_i \) to be a best response to \( \Delta^*_i \) regardless of the strategies of other agents, i.e. non-pure strategies are always dominated by a pure strategy. Due to symmetry, this applies to all agents in \( \mathcal{N} \). Therefore, it follows that a pure strategy NE always exists.

**APPENDIX B**

**PROOF OF PROPOSITION 1**

If the component \( C \) is not minimally connected, then it has at least one cycle as there exist agents \( i \) and \( j \) that are connected via two paths \( p_{ij,1} \) and \( p_{ij,2} \), such that any of the two paths is not a subset of the other. For such component at NE, assume that agent \( v \) is on path \( p_{ij,1} \) and agent \( w \) is on path \( p_{ij,2} \). Note that all the agents receive the same amount of total information \( H(C) \). We know that there indeed exists links: \( g^*_v \) (or \( g^*_w \)) and \( g^*_w \) (or \( g^*_y \)), where agent \( x \in p_{ij,1} \) and agent \( y \in p_{ij,2} \). Now focus on any link of them, say \( g^*_w = 1 \). We observe that agent \( w \) can break this link and still receive the same benefit by gathering the same amount of information from path \( p_{ij,1} \), thus receiving a strictly higher utility function as it will not pay the cost for the link with agent \( y \), which contradicts the fact that \( g^* \) is an NE. Thus, a single path exists between any two agents.

**APPENDIX C**

**PROOF OF LEMMA 1**

If there exists an agent in which other agents have an incentive to connect to even if they possess all other information in the network, then the network is indeed connected at any equilibrium. This is satisfied if and only if the linking cost satisfies \( c < f(H(X)) - f(H(X_i)) \) for some agent \( i \) in \( \mathcal{N} \), i.e. the marginal benefit from connecting to that agent is always more than the link cost irrespective to the current connections of the agent forming the link. Thus, we must have \( c < \max_i f(H(X_i)) - f(H(X_i)) \). Hence, part (i) of the Lemma follows.

If no agent have an incentive to form any link, then the network is fully disconnected. From the monotonicity property of the entropy, we know that if agent \( i \) has no incentive to connect to a set \( \mathcal{V} \) of agents, then it has no incentive to connect to a set \( \mathcal{U} \) if \( \mathcal{U} \subseteq \mathcal{V} \). Thus, if agent \( i \) has no incentive to connect to the set \( \mathcal{N}/\{i\} \) via a single link, then it has no incentive to form any link in the network. This occurs if \( c > f(H(X)) - f(H(X_i)) \). If this condition is satisfied for all agents, then the network is indeed disconnected, and part (ii) of the Lemma follows.
APPENDIX D

PROOF OF THEOREM 2

For the network to be in NE, no agent should have an incentive to unilaterally deviate by forming a new link or breaking a link. Focus on a certain component \( C_i \). Inside this component, each agent should either have an incentive to form at least one link, or other agents should have an incentive to connect to it. Otherwise, this agent can be removed from the component while strictly increasing the utility of some agent. Thus, we must have either \( f(H(X_{C_i})) - f(H(X_j)) > c \) or \( f(H(X_{C_i})) - f(H(X_{C_i}/\{j\})) > c \) for all agents \( j \) in \( C_i \). This should apply to all components in the network. Hence, condition (1) follows.

Now focus on the interaction between different components of the network. If any agent in component \( C_i \) benefits from forming a link to any agent in component \( C_j \), then the network is not NE since in this case an agent in \( C_i \) can strictly increase its utility by unilateral deviation. Hence, we should have \( f(H(X_{C_i}, C_j))) - f(H(X_{C_i})) \leq c \) for any two components in the network. Thus, condition (2) follows.

APPENDIX E

PROOF OF LEMMA 2

We know that in the \( \mathcal{K}_C \) region, all the NE networks are connected. Thus, the social welfare of any network in \( \mathcal{K}_C \) is given by \( U(g^*) = Nf(H(X')) - (N - 1)c \). The socially optimal network in \( \mathcal{K}_C \) is the one with a social welfare of \( \tilde{U} \), where \( \tilde{U} = \max_{g \in G} U(g) \). Since \( f(H(X')) - f(H(X_i)) > c, \forall i \) in the \( \mathcal{K}_C \) region, then it is clear that a connected network maximizes the social welfare. Therefore, \( \tilde{U} = U(g^*) \) and the PoA = 1 in the \( \mathcal{K}_C \) region.

Next, we focus on the \( \mathcal{K}_I \) region. In this region, any connection will result a negative payoff for any agent who forms a link since \( c > f(H(X')) - f(\min_i H(X_i)) \). Thus, the social optimal is a fully disconnected network, which is also the unique (strict) NE, and the PoA = 1 in the \( \mathcal{K}_I \) region. For the \( \mathcal{K}_M \) region, the maximum PoA will occur if a fully disconnected network is an equilibrium and a connected network is a social optimum. In what follows, we show that this is indeed possible. Consider the case when \( f(H(X_i, X_j)) - f(H(X_i)) < c, \forall i, j \), and \( f(H(X')) - f(H(X_i)) > c, \forall i \). In this case, agents do not get immediate benefit from forming links to individual agents, thus a fully disconnected network is an NE since not forming a link is a best response for all agents when all other agents do not form a link. Therefore, the PoA in the \( \mathcal{K}_M \) region is upper bounded by the social welfare of a connected network and that of a fully disconnected network, i.e. \( \text{PoA} \leq \frac{Nf(H(X)) - (N - 1)c}{\Sigma_{i=1}^N f(H(X_i))} \).

APPENDIX F

PROOF OF THEOREM 4

The PoA can be written as \( \text{PoA} \leq \frac{Nf(\Sigma_{i=1}^N H(X_i)) - D(p|q) - (N - 1)c}{\Sigma_{i=1}^N f(H(X_i))} \). Note that the benefit function \( f(x) \) is monotonically increasing in \( x \). Thus, as \( D(p|q) \) increases, \( f(\Sigma_{i=1}^N H(X_i)) - D(p|q) \) decreases, and the PoA decreases consequently. Therefore, we have \( \frac{\partial \text{PoA}}{\partial D(p|q)} < 0 \).
APPENDIX G

PROOF OF COROLLARY 1

In the $K_C$ region, we know that all NE networks are connected. Thus, $\sup_{\tilde{g}^* \in G^c} H(X_i \cup X_{R_s, g^*_i}) = H(\mathcal{X})$, and MIL $= 0$. Similarly, in the $K_I$ region, we have $\sup_{\tilde{g}^* \in G^c} H(X_i \cup X_{R_s, g^*_i}) = \inf_{\tilde{g}^* \in G^c} H(X_i \cup X_{R_s, g^*_i}) = \min_i H(X_i)$, thus MIL $= 0$. In the $K_M$ region, the MIL is maximized if both a connected and a fully disconnected network are equilibria. In this case, $\sup_{\tilde{g}^* \in G^c} H(X_i \cup X_{R_s, g^*_i}) = H(\mathcal{X})$, and $\inf_{\tilde{g}^* \in G^c} H(X_i \cup X_{R_s, g^*_i}) = \min_i H(X_i)$. Thus, MIL \leq H(\mathcal{X}) - \min_i H(X_i).

APPENDIX H

PROOF OF COROLLARY 2

From Theorem 5, we know that when $c > k\bar{H}$, then we have a unique equilibrium $s^*$ for both $\tilde{F}_H$ and $\tilde{E}_H$ in which $g^*_i = 0$, $\forall i, j \in \mathcal{N}$, and $H^*(X_i) = \bar{H}$. Thus, we have $H^*(X_i) > 0, \forall i \in \mathcal{N}$, and $\frac{|\mathcal{I}(s^*)|}{N} = 1$, which applies when the number of agents in the CIN grows to infinity, hence (9) follows. Next, we focus on the total amount of information in the network. For $\tilde{E}_H$, we have $H(X_1, X_2, ..., X_N) = \max\{\bar{H}, \bar{H}, ..., \bar{H}\} = \bar{H}$, and (10) follows. Finally, for $\tilde{F}_H$, we have $H(X_1, X_2, ..., X_N) = \sum_{i=1}^{N} \bar{H} = NH$, and (11) follows.

APPENDIX I

PROOF OF COROLLARY 3

We start by deriving (12). From Theorem 6, we know that for $\tilde{F}_H$, every equilibrium has only one information producer. When the number of agents grows to infinity, we will still have one information producer and $\frac{|\mathcal{I}(s)|}{N} = 0$. In order to prove (13), one needs to find one network in equilibrium for $\tilde{F}_H$ in which, for arbitrary $N$, we have $N$ information producers. Consider this network for $N$ agents. Assume that $H(X_i) = \frac{\bar{H}}{N}, \forall i \in \mathcal{N}$, and the network has a single component which is periphery-sponsored star network. It is clear that for this network, $|\mathcal{I}(s)| = N$. We want to show that this network is an NE by showing that every agents strategy is best response to all others. It is easy to see that since $c < k\bar{H}$, each periphery agent has no incentive to break its link with the core since $\frac{N-1}{N}k\bar{H} > c$ when $N$ is asymptotically large. Moreover, no agent has incentive to alter its information production profile since the total information in the network is $\sum_{i=1}^{N} \frac{\bar{H}}{N} = \bar{H}$. Thus, $s$ is an NE. Since this applies to any $N$, (13) follows. Finally, since the network is always connected in any equilibrium, then (14) directly follows.

REFERENCES


