

# The population dynamics of websites

[Online Report]

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## ABSTRACT

Websites derive revenue by advertising or charging fees for services and so their profit depends on their user base – the number of users visiting the website. But how should websites control their user base? This paper is the first to address and answer this question. It builds a model in which, starting from an initial user base, the website controls the growth of the population by choosing the intensity of referrals and targeted ads to potential users. A larger population provides more profit to the website, but building a larger population through referrals and targeted ads is costly; the optimal policy must therefore balance the marginal benefit of adding users against the marginal cost of referrals and targeted ads. The nature of the optimal policy depends on a number of factors. Most obvious is the initial user base; websites starting with a small initial population should offer many referrals and targeted ads at the beginning, but then decrease referrals and targeted ads over time. Less obvious factors are the type of website and the typical length of time users remain on the site: the optimal policy for a website that generates most of its revenue from a core group of users who remain on the site for a long time – e.g., mobile and online gaming sites – should be more aggressive and protective of its user base than that of a website whose revenue is more uniformly distributed across users who remain on the site only briefly. When arrivals and exits are stochastic, the optimal policy is more aggressive – offering more referrals and targeted ads.

## Categories and Subject Descriptors

J.4 [Social and Behavioral Sciences]

## Keywords

Websites, referrals, population dynamics

## 1. INTRODUCTION

Many popular websites such as Facebook, Google, and Netflix<sup>1</sup> derive a significant portion of their revenue through advertising or by charging subscription fees to their users<sup>2</sup>. Given such a revenue model it is critical for the websites to obtain and maintain a healthy user base. Hence, a critical question that needs to be answered by such websites is: how can the websites control their user base? The user

traffic on a website comes from two channels: sponsored and non-sponsored. Sponsored traffic is steered to the website through targeted ads, referrals, paid keywords, discounts etc. Hence, the website needs to decide how aggressively it should advertise as well as send referrals to control the user base given the incurred costs.

This paper aims to study and design policies which can be deployed by websites to control their user base through referrals and targeted ads in order to maximize their profits. Among the works on user base dynamics in the economics literature, the ones that relate the most to this work are [5] and [3]. In [5] the authors analyze the effects of search frictions in building a customer base on the firm's profits, investment, sales, etc. In [3] the authors study informative advertising and analyze the effect of a decline in the cost of information dissemination on the firm and customer dynamics. The key results in [5][3] are derived by calibrating models based on data, while this work instead provides a theoretical foundation for understanding user base dynamics. Also, the focus of these works [5][3] are on a firm selling a product to homogeneous users; in contrast the users on the website in our model can be heterogeneous and generate varying revenue depending on the number of advertisements they click.

We propose a dynamic continuous time model for the population of users present on a website. We assume that the website starts with a small initial user base and at every moment in time, the website can reach out to potential users via referrals and targeted ads. The website does this by paying a cost to incentivize its current users to send referrals to friends or by paying for targeted ads on other platforms. Thus, the website must adopt a policy that balances the marginal benefit of increasing its user base versus the marginal cost of providing the referrals and targeted ads. Our model accounts for the fact that users are heterogeneous, and therefore provide different benefits to the website<sup>3</sup>. We model this by making the natural assumption that the benefit function of the website is concave in the population level: the value from adding new users decreases with an increasing population level.

If the website starts with a low initial population, we show that the optimal policy will be to give many referrals per unit time initially and then decrease the referrals over time. This

<sup>1</sup>See <http://www.pewinternet.org/2015/01/09/social-media-update-2014/> and <http://www.pewglobal.org/2012/12/12/social-networking-popular-across-globe/>.

<sup>2</sup>See <http://finance.yahoo.com/news/must-know-assessing-facebook-revenue-170009607.html>

<sup>3</sup>For instance, in mobile gaming apps half of the revenue coming from only 0.15% of the user base. See <http://venturebeat.com/2014/02/26/only-0-15-of-mobile-gamers-account-for-50-percent-of-all-in-game-revenue-exclusive/>

is applicable for most websites that are new to the market and are unlikely to be able to acquire a high initial population. We also analyze how the website's optimal policy depends on the average time that a user stays on the website before exiting. We show that this depends critically on the revenue distribution of the website across its users. For instance, some websites rely heavily on a small set of users for revenue, e.g. mobile and online gaming websites, which rely on a small group of players who spend a lot on games. For such websites the optimal policy will increase the referrals and targeted ads if the average stay time of the users decreases. On the other hand, some websites obtain revenue from its users more uniformly, e.g. subscription based websites such as Netflix. For such websites the optimal policy will become more aggressive – to send more referrals and ads if the average time that a user stays on the website increases. We also extend our model to allow for stochastic user arrivals/exits, and we show that the above results are robust: if there is high uncertainty in user arrivals, we show that the website's policy will be to send more referrals and ads on average.

## 2. MODEL

We assume that there is a continuum of potential users and the firm can attract these users by posting ads on search engines (Google) and other websites (Facebook) or by giving referrals. New users can visit the website either through such sponsored media or arrive directly through exogenous methods, such as arriving upon the website through a non-sponsored link. We assume that the rate of such arrivals is a constant  $\theta$ . This is a simplifying assumption made due to page limitations and the key results of this work continue to hold even under more general arrival rates that depend on the population levels such as  $\theta \cdot p^s$  with  $0 \leq s \leq 1$ . The users are heterogeneous, i.e. the revenue that the users generate for the website (e.g., the number of ads a user clicks) varies across the users. Every user who visits the website stops using the website after a random time. This time is a random variable drawn from an exponential distribution and the average time a user stays is  $\eta$ . Denote the total mass of the users using the website at time  $t$  as  $p(t)$ , where  $p : [0, \infty) \rightarrow [0, \infty)$ . The initial user base is denoted as  $p(0)$ . This total mass of the users represents the unique user statistic, which is often used as a metric to evaluate a website's popularity<sup>4</sup>.

The revenue that the website generates per unit time increases with the mass of the users. It is denoted as  $b(p)$ , where  $b : \mathbb{R} \rightarrow [0, \infty)$  is a continuously differentiable increasing function. Moreover, as mentioned in the introduction  $b(p)$  is assumed to be a concave function in  $p$ . The long-term benefit of the website considering a discount rate of  $\rho$  can be computed as  $B(p(\cdot)) = \int_0^\infty b(p(t))e^{-\rho t} dt$ . The website chooses the intensity of advertisements (measured in terms of number of sponsored links, referrals, targeted ads) to be posted on other platforms at each time  $t$  and as a result it controls the rate of sponsored arrivals  $\lambda(t)$ , where  $\lambda : [0, \infty) \rightarrow \mathbb{R}_+$  is a continuous and bounded function. The rate of change of population on the website is  $\frac{dp}{dt} = \theta + \lambda(t) - \frac{1}{\eta}p$ . The first two terms in the differential equation represent the rate of direct and sponsored arrivals respectively, while the third term represents the users ex-

iting the website. The firm bears a cost per unit time for the advertisements, which increases with the sponsored user arrivals  $\lambda(t)$ . It is given by  $c(\lambda)$ , where  $c : \mathbb{R} \rightarrow [0, \infty)$  is a continuously differentiable increasing function. Given the heterogeneous pool of potential users it is harder to increase the intensity of arrivals when  $\lambda$  is large. Hence, we assume that the cost  $c(\lambda)$  is strictly convex in  $\lambda$  (as in [5]). We also assume that there is no cost when there are no sponsored arrivals  $c(0) = 0$ , and that the cost becomes unbounded as the sponsored arrivals approach  $\infty$ , i.e.  $\lim_{\lambda \rightarrow \infty} c(\lambda) = \infty$ . The long-term discounted average cost to the website is  $C(\lambda(\cdot)) = \int_0^\infty c(\lambda(t))e^{-\rho t} dt$ .

The firm desires to maximize its total discounted profit, i.e.  $B(p(\cdot)) - C(\lambda(\cdot))$ . The continuous-time optimization problem of the firm is stated as follows

$$\begin{aligned} & \max_{\lambda(t) \in \mathbb{R}_+, \forall t \geq 0} B(p(\cdot)) - C(\lambda(\cdot)) \\ & \text{subject to } \frac{dp}{dt} = \theta + \lambda(t) - \frac{1}{\eta}p(t), p(0) \text{ is given} \end{aligned}$$

Next, we analyze the behavior of the optimal policy.

## 3. RESULTS

We can show that the optimal policy exists (See [1] for details) and is denoted by  $\lambda_{p(0)}(t)$ . Denote the corresponding population dynamic as  $p_{p(0)}(t)$ . If the optimal policy and the corresponding population dynamic converge, the steady state is achieved. The next theorem provides conditions for the existence of the steady state and also establishes that the steady state is unique. This theorem uses the the steady state population level  $\hat{p}$ , which is given by

$$X(\hat{p}) = b'(\hat{p}) \frac{1}{\rho + \frac{1}{\eta}} - c'(\frac{1}{\eta}\hat{p} - \theta) = 0 \quad (1)$$

with  $b'(p) = \frac{db(p)}{dp}$  and  $c'(\lambda) = \frac{dc(\lambda)}{d\lambda}$ .

**THEOREM 1. Steady state: Existence and Uniqueness**

- i) If  $0 < b'(0) < \infty$  and  $\lim_{p \rightarrow \infty} X(p) < 0$  then there exists a unique solution  $\hat{p}$  to the steady state equation (1).
- ii) If  $p(0) < \hat{p}$  then the optimal policy  $\lambda_{p(0)}(t)$  decreases with time and converges to  $\hat{\lambda} = \frac{1}{\eta}\hat{p} - \theta$  and the corresponding population dynamic  $p_{p(0)}(t)$  increases and converges to  $\hat{p}$ .

For the proofs of all the theorems refer to the online version [1]. Theorem 1 proves that if there is a positive marginal benefit from increasing the user base at very low population levels and if there is a negative marginal benefit from increasing the user base at very high populations then there exists a population level (between very low and very high population levels) where the marginal benefit is zero, which corresponds to the unique steady state. In addition we see that if the initial population level is low, which is true for most of the websites when they are launched, then the firm is more aggressive with advertising in the initial stages (closer to the launch) in comparison to the later stages (closer to the steady state). For the rest of the paper it is assumed that the firm starts with a low initial population, i.e.  $p(0) < \hat{p}$ . Also, we will use the terms advertisements and referrals, firm and website interchangeably henceforth.

### 3.1 Policy for Different User Behaviors

There are several interesting questions that one can ask about how the optimal policy depends on the user's behavior: What happens to the policy if the average time that a

<sup>4</sup><http://www.pcmag.com/encyclopedia/term/53438/unique-visitors>

user stays on the website decreases? Or if the direct arrivals to the website increase? How does the new policy compare with the old policy both in the steady state (i.e.  $t \rightarrow \infty$ ) and at a finite time  $t$ ? We first address these questions at time  $t \rightarrow \infty$ , i.e. in the steady state.

### 3.1.1 Comparison of the policy in the steady state

If the average stay time of the users decreases then the firm is faced with the following question: Is spending more on advertisements worth it? The answer is not straightforward because of the following dilemma. The average stay time of the users reduces which discourages the firm, but the users leaving at a faster rate will also reduce the population and the total benefit may thus fall sharply. We answer this question next.

Define an operator  $\Phi_{\theta,\eta,\rho}(b(p), p) = \frac{d^2 b(p)}{dp^2} p\eta + \frac{db(p)}{dp} \frac{1}{\rho + \frac{1}{\eta}}$ .

The next theorem shows that if  $\eta$  decreases and  $\Phi_{\theta,\eta,\rho}(b(\hat{p}), \hat{p}) < 0$  then the intensity of ads increases, otherwise the intensity decreases.

**THEOREM 2.** *Policy comparison with change in  $\eta$  in steady state  $t \rightarrow \infty$ : Local Behavior*

i) *If the average time that a user stays on the website  $\eta$  is decreased by  $\epsilon > 0$  and if  $\Phi_{\theta,\eta,\rho}(b(\hat{p}), \hat{p}) < 0$  then the advertisements in the steady state increase.*

ii) *If the average time that a user stays on the website  $\eta$  is decreased by  $\epsilon > 0$  and if  $\Phi_{\theta,\eta,\rho}(b(\hat{p}), \hat{p}) > 0$  then the advertisements in the steady state decrease.*

We assume that the change in  $\eta$ , i.e.  $\epsilon$  to be small (See [1] for details). Theorem 2 is interpreted as follows. If  $\Phi_{\theta,\eta,\rho}(b(\hat{p}), \hat{p}) < 0$  then the percentage increase in the marginal benefit resulting from a decrease in the user population is

high, i.e.  $\frac{\frac{d^2 b(p)}{dp^2}|_{p=\hat{p}}}{\frac{db(p)}{dp}|_{p=\hat{p}}} < -\frac{1}{(\rho + \frac{1}{\eta})\hat{p}\eta}$ . Therefore the website should send out more ads if the average stay time reduces. Conversely, if  $\Phi_{\theta,\eta,\rho}(b(\hat{p}), \hat{p}) > 0$  then the percentage increase in marginal benefit resulting from a decrease in population is not sufficient to compensate for the marginal cost of additional ads. Therefore, the firm reduces the ads.

The above theorem characterizes the local behavior of the firm's policy under the impact of small changes in the average stay time. Next, we characterize how the firm's global behavior under the impact of arbitrary changes in average stay time. We focus on a specific benefit function for better exposition, while the results presented extend to a larger class of functions. Consider a benefit function defined for  $p \geq 0$ , as  $b(p) = p^a$  with  $0 < a < 1$ . For a fixed  $a$ , we define a threshold  $\bar{\eta} = \frac{a}{\rho(1-a)}$ . Then the following is true for the behavior of the firm.

**RESULT 1.** *Policy comparison with change in  $\eta$  in steady state  $t \rightarrow \infty$ : Global Behavior*

*If the average stay time of the users decreases upto  $\bar{\eta}$  then the firm should increase the intensity of advertisements in the steady state, while if the average stay time falls below  $\bar{\eta}$  then the firm should decrease the advertisements.*

Next, we analyze the firm's behavior depending on its revenue distribution across the users, which is reflected by  $a$ . If  $a$  is small then the firm's benefit saturate very fast. This occurs if the firm relies heavily on a set of core users for revenue. Example of such websites are online and mobile

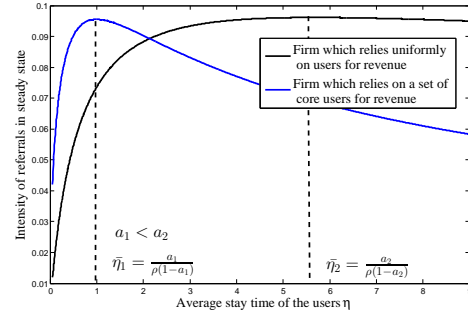


Figure 1: Impact of average stay time on firm's behavior in steady state

gaming, since these websites rely heavily on their core users. Observe that a small value of  $a$  implies that the threshold  $\bar{\eta}$  is very small as well. Hence, for such firms a reduction in the average stay time leads to an increase in the advertisements in the steady state. On the other hand, consider the case when  $a$  is large and close to 1. If  $a$  is close to 1 then the firm's benefit saturates slowly. This occurs if the firm relies uniformly on all the users for revenue. Examples include subscription based websites such as Netflix. Observe that a large value of  $a$  implies a high threshold  $\bar{\eta}$ . Hence, for such firms an increase (decrease) in the average stay time leads to a increase (decrease) in the advertisements in the steady state. As an example, the increase in the content quality and thus user stay time on Netflix led Netflix spending on expanding its user base <sup>5</sup>. Fig. 1 shows the example of the above two types of websites. Next, we understand the effect of an increase in the direct arrivals on the intensity of advertisements in steady state.

**THEOREM 3.** *Policy comparison with change in  $\theta$  in steady state  $t \rightarrow \infty$ : Global Behavior*

*If the intensity of the direct arrivals  $\theta$  is increased then the intensity of advertisements in the steady state decreases.*

The intuition behind Theorem 3 is as follows. An increase in the direct arrivals makes the firm decrease the advertisements in a controlled manner, such that the total population on the website for the new higher level of direct arrivals is higher than the case with original lower level. Hence, the firm can reduce its cost while simultaneously increasing the total population.

We have analyzed the impact of the revenue distribution of the firm and the average stay time of the users on the firm's policy in steady state. Now we want to understand the impact on the policy while it is on the path to steady state. For this, we will assume quadratic benefit and cost functions from now on due to space limitations. However, the results obtained extend to a larger class of benefit and cost functions.

### 3.1.2 Comparison of the policy on the path to steady state

Websites are hosted on servers and the traffic that can be handled by a server is limited. Hence, we assume that the website has a maximum capacity, which corresponds to the maximum mass of users that the website can support with

<sup>5</sup><http://www.forbes.com/sites/petercohan/2015/01/21/4-reasons-to-invest-in-netflix/>

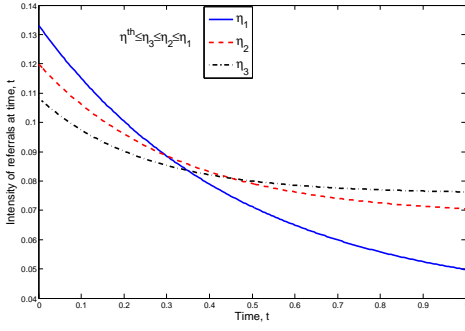


Figure 2: Impact of average stay time of the user  $\eta$  on firm's behavior out of the steady state when  $\eta \geq \eta^{th}$

a non-negative marginal benefit. The benefit per unit time is a continuously differentiable concave function  $b: \mathbb{R}_+ \rightarrow \mathbb{R}$  defined as  $b(p) = \gamma^2 - (\gamma - p)^2$ , where  $\gamma$  is the capacity of the website. In the previous subsection the benefit function was increasing in  $p$ , while this is no longer the case when capacity is incorporated in the model. We will assume that the capacity is sufficiently large, which will ensure that the results of the previous section continue to hold. We discuss the impact of firm choosing to invest in increasing the capacity in the online version of the paper [1]. We also assume that the cost per unit time for sponsored arrivals is quadratic (as in [5]) and is given as  $c(\lambda) = c\lambda^2$ .

We will assume through out this section that the capacity is sufficiently high, i.e.  $\gamma \geq \theta\eta$ . This condition ensures that if there are no advertisements given at any time then the population in the steady state is less than the capacity. Under this assumption we can show that the result in Theorem 1, 2 and, 3 continue to hold. Hence, in the optimal policy the advertisements will decrease with time and achieve the steady state value  $\hat{\lambda}$  (from Theorem 1). If the average stay time of the users were to decrease then the firm will continue to increase the advertisements  $\hat{\lambda}$  until the stay time falls below a threshold, beyond which the advertisements decrease. Next, we analyze the effect of a decreased stay time on the policy outside the steady state.

**THEOREM 4. Policy comparison with change in  $\eta$ : at finite time  $t$**

i) If the average stay time of the user decreases to a value above the threshold  $\eta^{th}$  then the intensity of advertisements increases at all time instances  $t$  larger than some finite  $t^*$ .

ii) If the average stay time of the user  $\eta$  decreases to a value below the threshold  $\eta^{th}$  then the intensity of advertisements at every time instance  $t$  decreases.

See the expression for  $\eta^{th}$  and  $t^*$  in the online version [1]. This theorem generalizes the result in subsection 2.2.1., where the behavior was analyzed in the steady state. In Fig. 2 we can see that if the average stay time of the user  $\eta$  reduces but does not fall below  $\eta^{th}$  then the firm spends more on advertisements – this would happen when the website is sufficiently old, i.e.  $t \geq t^*$ . This behavior from an incumbent website could serve as a barrier to entry to an entrant website, whose arrival possibly results in the decrease in the average stay time. Fig. 3 shows the second case, i.e. if the average stay time of the user decreases below the threshold  $\eta^{th}$  the website gives out less referrals in the new optimal policy.

Next, we understand the effect of an increase in the direct

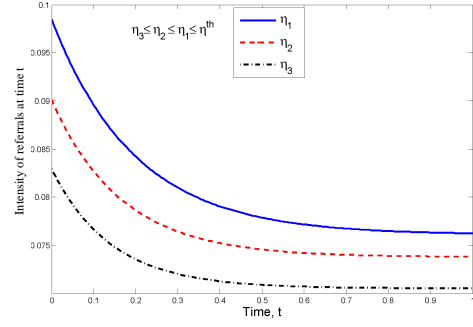


Figure 3: Impact of average stay time of the user  $\eta$  on firm's behavior out of the steady state when  $\eta \leq \eta^{th}$

arrivals on the policy outside the steady state. It can be shown that there will be a decrease in the advertisements at all times (See [1]). This generalizes Theorem 3. The intuition for this result is the same as that for Theorem 3.

**Uncertain user arrivals/exits:** We can show that even under uncertain arrivals/exit of the users the key results presented in the deterministic setting will continue to hold (See [1]). The population dynamic with uncertain user arrival/exit changes into the following stochastic differential equation  $dP = (\theta + \lambda(t) - \frac{1}{\eta}P)dt + \sigma P dW_t$ . Here  $W_t$  is the Weiner process/Brownian motion. The firm's objective in this case is to maximize the expectation of the long term total discounted profit (benefit-cost). If we consider the quadratic benefit and costs considered in the previous section, Theorem 4 can be extended to the stochastic setting as well (See [1]). In this case the comparison is done between the expectation of the intensity of sponsored arrivals. An interesting result emerges is when we compare the intensity of the advertisements under varying levels of uncertainty  $\sigma$ . It can be shown that increasing the level of uncertainty makes the optimal policy more aggressive in giving advertisements (See [1]).

## 4. CONCLUSION

This paper has been the first to systematically characterize how a website should build its user base. We showed that the optimal policy requires that websites starting with small initial populations send ads aggressively in the beginning and then decrease them with time. For websites that derive revenue from a set of core users, e.g. mobile and online gaming, the optimal policy increases ads when the average stay time of the user on the site decreases, while for websites with a more uniform revenue distribution across users the optimal policy decreases ads. These results extend to a stochastic setting with noise in population dynamics, and it is shown that more uncertainty in user arrivals/exits leads to a more aggressive optimal policy for ads.

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## 6. PROOFS

Lemma 1. For the proposed model the optimal policy always exists.

PROOF. Existence of the optimal policy

We first show that the optimal policy  $\lambda(\cdot)$  exists. To be able to do so we will use the sufficient conditions arrived at in [4]. The search space of optimal policy  $\lambda(\cdot)$  is restricted to measurable functions and the optimal policy is positive uniformly bounded above by  $M$ . The population mass at time  $t$   $p(t)$  is restricted to be absolutely continuous function on interval  $[0, T]$  and this has to hold for all such intervals. The initial condition on  $p(t)$  is given, i.e.  $p(0)$ . The trajectory of  $p$  follows the following differential equation  $\frac{dp}{dt} = h_{\theta, \eta}(\lambda, p) = \theta + \lambda(t) - \frac{1}{\eta}p$ . We now show that the assumptions in [4] hold for our model as well.

A1. The function  $h_{\theta, \eta}(\lambda, p)$  is linear in  $p$  and is independent of time  $t$ , therefore clearly the function is continuous in  $(t, p)$ . The function is also linear in  $\lambda$ .

Let  $S(t) = [-\delta, P]$  denote the set that the population at time  $t$  is restricted to be in, where  $\delta > 0$ . In our original problem we did not have such a set to restrict the population trajectory. Hence, imposing such a constraint may not lead to an equivalent problem. Note that if  $U$  is chosen sufficiently high then the two problems are equivalent. Observe that if the initial population level is  $p(0)$  then any trajectory of  $p(t)$  which follows  $\frac{dp}{dt} = h_{\theta, \eta}(\lambda, p)$  will always be bounded above by  $P = \max\{p(0), (\theta + U)\eta\} + \delta$ . This can be justified as follows. Consider the case when  $p(0) < (\theta + U)\eta$ . Let us assume that the population  $p(t)$  in this case does exceed  $P$ , which is equal to  $(\theta + U)\eta + \delta$ . Since  $p(t)$  has to exceed  $(\theta + U)\eta$  it has to be true that there exists a  $t'$  at which  $p(t) = (\theta + U)\eta$  and  $\frac{dp}{dt}|_{t=t'} > 0$ . This is due to the continuity of  $p(\cdot)$ . However, notice that  $h_{\theta, \eta}(\lambda, (\theta + U)\eta) = \theta + \lambda(t) - \frac{1}{\eta}p$  is non-positive because  $\lambda \leq U$ . This contradicts the claim. Consider the case when  $p(0) > (\theta + U)\eta$ . In this case we need to show that the population trajectory  $p(t)$  always stays less than or equal to  $p(0)$ . Let us assume otherwise, i.e. the population did exceed  $p(0)$ . It can be shown that for the population to exceed and attain a level higher than  $p(0)$ , the rate of change of population  $\frac{dp}{dt}$  has to be positive at a certain time when  $p(t) = p(0)$ . This cannot be true because  $\lambda \leq U$  which will make the  $\frac{dp}{dt} = h_{\theta, \eta}(\lambda, (\theta + U)\eta) < 0$ . Note that in the above case we arrived at a bound  $P$ , which depends on  $p(0)$ . We can make this bound independent of  $p(0)$  by assuming that the initial population  $p(0) < P_{in} < \infty$ .

Therefore, if  $S \geq P$  then the problem with constraint on  $p(t) \in S(t)$  will still be equivalent.

A2. Since  $S(t)$  is closed for all  $t$  the second assumption in [4] is satisfied as well.

A3. Since  $p(0)$  is fixed then assumption 3 is automatically satisfied.

Let  $U(t, p)$  denote the set in which the  $\lambda(t)$  is restricted

to be in. In our case let us assume that  $U(t, p) = [0, U]$ . Since  $\lambda(\cdot)$  is upper bounded by  $U$  this implicit constraint suffices for the equivalence of the two problems if we add the constraint that  $\lambda(t) \in U(t, p)$ .

A4. Since the set valued function  $U(t, p) = [0, U]$  is closed and bounded and takes a fixed value, we can deduce that it is a continuous mapping and the output of the mapping  $U(t, p)$  is convex and compact.

A5. Since the set  $U(t, p)$  and  $S(t)$  are uniformly bounded then we can say that the assumption 5 is satisfied.

A6. The function  $\beta(p(0)) = 0$  in our case and hence, continuity is obvious.

A7.  $\phi(t, p, \lambda) = (c(\lambda) - b(p))e^{-\rho t}$  is continuous in  $(t, \lambda, p)$ . This follows from the definition of the functions  $b(\cdot)$  and  $c(\cdot)$ . Also the convexity  $c(\cdot)$  is sufficient to show that  $\phi(t, p, \lambda)$  is convex in  $\lambda$ .

A8. Consider the negative part of the function  $\phi(t, p, \lambda) = (c(\lambda) - b(p))e^{-\rho t}$ , which is equal to  $-b(p)e^{-\rho t}$  and we need to show that negative part of  $\phi(t, p, \lambda)$  i.e.  $\lim_{T' \rightarrow \infty} \int_{T'}^{T''} -b(p(t))e^{-\rho t} dt$  goes to zero. Observe that  $0 \geq \int_{T'}^{T''} -b(p(t))e^{-\rho t} dt \geq \int_{T'}^{\infty} -b(P)e^{-\rho t} dt = -b(P)e^{-\rho T'} \frac{1}{\rho}$ . Hence, clearly the limit of the term goes to 0. Note that this limit will be 0 for all population trajectories.

In addition we also know that there is a feasible trajectory i.e. which satisfies all the conditions above and gives a finite value of the objective, consider the case when the  $\lambda(t) = 0$  then  $V(p(0)) = \int_0^{\infty} b(\theta\eta(1 - e^{-\frac{t}{\eta}}) + p(0)e^{-\frac{t}{\eta}})e^{-\rho t} dt$ . Since  $b(\theta\eta(1 - e^{-\frac{t}{\eta}}) + p(0)e^{-\frac{t}{\eta}}) \leq b(P)$  we know that the integral exists.

Given all the conditions in [4] are satisfied the optimal policy has to exist.

□

Theorem 1. Steady state: Existence and Uniqueness

i) If  $0 < b'(0) < \infty$  and  $\lim_{p \rightarrow \infty} X(p) < 0$  then there exists a unique solution  $\hat{p}$  to the steady state equation (1).

ii) If  $p(0) < \hat{p}$  then the optimal policy  $\lambda_{p(0)}(t)$  decreases with time and converges to  $\hat{\lambda} = \frac{1}{\eta}\hat{p} - \theta$  and the corresponding population dynamic  $p_{p(0)}(t)$  increases and converges to  $\hat{p}$ .

PROOF. In the steady state  $\frac{dp}{dt} = 0$  which implies  $\lambda = \frac{p}{\eta} - \theta$ . In order to arrive at the steady state solution we would use the HJB equation, which is a necessary condition which the optimal value function needs to satisfy. It is given as follows.

$$\rho V(p) = \max_{\lambda \geq 0} (b(p) - c(\lambda) + V'(p)(\theta + \lambda - \lambda_a p))$$

Note that the above equation assumes that the value function is differentiable. Next, we show that the optimal value function is differentiable in our problem. In [2] the authors showed sufficient conditions on the problem that are required for a value function to be differentiable at a point. Define the set in which the  $(p(t), \frac{dp}{dt})$  are constrained to be in.  $p(t)$  is constrained to be in the set  $S$  and  $\frac{dp}{dt}$  is constrained to be in  $[\theta - \frac{P}{\eta} - \delta, \theta + U + \delta]$ . The upper bound on  $\frac{dp}{dt}$  comes from the constraint that  $\lambda \leq U$  and the population is lower bounded by 0. While the lower bound follows from the fact that  $p(t) \leq P$ . Denote the set in which  $(p(t), \frac{dp}{dt})$  are constrained to be in as  $T$ . Hence  $T$  is given as  $T = S \times (-\infty, \theta + U]$

A1. The first assumption requires the set  $T$  to be convex and have a non-empty interior. Clearly the set  $T$  is convex and has a non-empty interior.

A2. Let the flow benefit function be given as  $u(p, \frac{dp}{dt}, t) = (b(p) - c(\lambda))e^{-\rho t} = (b(p) - c(\frac{dp}{dt} + \frac{p}{\eta} - \theta))e^{-\rho t}$ . Note that  $b(p)$  is concave and  $-c(p + \frac{dp}{dt})$  is concave in  $(p, \frac{dp}{dt})$ . Hence, the sum of both the functions has to be concave in  $(p, \frac{dp}{dt})$ . Also,  $b(p)e^{-\rho t}$  is continuously differentiable. This is because we assume that  $b(p)$  is continuously differentiable and  $e^{-\rho t}$  is continuously differentiable in  $t$ . Similar argument holds for the other part of the function  $c(\frac{dp}{dt} + \frac{p}{\eta} - \theta)e^{-\rho t}$ . Hence, the flow benefit function is continuously differentiable. Therefore, the second assumption is satisfied.

A3. The optimal policy  $p(\cdot), \lambda(\cdot)$  exists starting from  $p(0)$ . This follows from the Lemma 1, where we showed that the optimal solution exists. The previous lemma holds for any value of  $p(0) < \infty$ , hence the optimal policy exists for all  $p(0) < \infty$  and therefore  $V(p)$  is well defined. Note that  $|V(p)| = |\int_0^\infty (b(p(t)) - c(\lambda(t)))e^{-\rho t} dt|$  exists and is bounded because  $p(\cdot)$  and  $\lambda$  are bounded.

A4. In this assumption it is required to show that the solution starting at  $p(0)$ , i.e.  $p_{p(0)}(t)$  and the corresponding  $\frac{dp_{p(0)}}{dt}$  are interior in the sense stated in [2]. Note that by the construction of the set  $S$  the entire trajectory of  $p_{p(0)}(t)$  is in the interior, because the  $p(t) > -\delta$  and  $p(t) < P$ . Hence, at no point does the trajectory hit the boundary, thereby showing that this assumption is true as well. Similar argument holds for the  $\frac{dp_{p(0)}}{dt}$  as well.

Given the above four assumptions, the value function is differentiable at  $p(0)$ . The above argument can be extended to any  $p(0)$  provided we assume that  $p(0) < P_{in}$ . This will ensure that the bound  $P$  can be now modified to  $\max\{P_{in}, (\theta + U)\eta\} + \delta$ .

The optimal solution to the HJB equation  $\lambda = c'^{-1}(V'(p))$ . This follows from FOC and the fact that  $c'$  is invertible and  $c'^{-1}(x)$  is positive when  $x > 0$ . This can be justified as follows.  $c$  is strictly convex, therefore  $c'$  is strictly increasing, which implies that  $c'$  is invertible. It is assumed that  $c'(0) = 0$  and  $\lim_{x \rightarrow \infty} c'(x) = \infty$ , using this and strictly increasing nature of  $c'$ , it is clear that  $c'^{-1}(x) > 0$  for  $x > 0$ . However, we also need to show that  $V'(p)$  is positive. This means that the value function is increasing with an increase in the initial population base, this seems intuitive owing to the increasing nature of the benefit function as a function of population. We show this below.

Let  $p_1 < p_2$  and let's assume  $V(p_1) > V(p_2)$ . Let the optimal policy starting at an initial population of  $p_1$  be  $\lambda_{p_1}(t)$ . Suppose we followed the same policy  $\lambda_{p_1}(t)$  at the initial population  $p_2$  the population levels achieved by the trajectory starting at  $p_2$  will always be greater than or equal to the trajectory at  $p_1$ . The trajectory starting at  $p_2$  has to be strictly greater for a finite duration, before which it intersects with the trajectory of  $p_1$ . After the time the two trajectories intersect the two trajectories follow the same path. It may also happen that the trajectory of  $p_2$  always stays higher. Since the population levels are always higher and the costs for the referrals is the same in the two cases, the value function at  $p_2$  is bound to be higher. This contradicts the assumption. Hence,  $p_1 < p_2$  implies  $V(p_1) < V(p_2)$ .

Hence, we know that in steady state the following has to

hold.

$$\lambda = \frac{\hat{p}}{\eta} - \theta$$

$$c'(\frac{\hat{p}}{\eta} - \theta) = V'(\hat{p})$$

Since  $V'(\hat{p})$  is not known the above equations are not sufficient to derive  $\hat{p}$ . Since we know that in steady state  $\frac{dp}{dt} = 0$ , therefore we know that  $V(\hat{p}) = \frac{b(\hat{p}) - c(\frac{\hat{p}}{\eta} - \theta)}{\rho}$ . Define another function  $W(p) = \frac{b(p) - c(\frac{p}{\eta} - \theta)}{\rho}$ . Observe that if the policy was  $\lambda = \frac{p}{\eta} - \theta$  then the corresponding  $\int_0^\infty (b(p(t)) - c(\lambda(t)))e^{-\rho t} dt = \frac{1}{\rho} (b(p) - c(\frac{p}{\eta} - \theta)) = W(p)$ . It is clear that since  $V(p)$  is the supremum over all the policies, the following has to hold  $V(p) \geq W(p)$ . And the equality holds at  $\hat{p}$ , i.e.  $V(\hat{p}) = W(\hat{p})$ . Since  $b(p)$  is concave and  $-c(\frac{p}{\eta} - \theta)$  is also concave in  $p$ . Therefore,  $W(p)$  is concave and continuously differentiable in  $p$ . We can conclude that  $V'(p) = W'(p) = \frac{b'(p) - \frac{1}{\eta}c'(\frac{p}{\eta} - \theta)}{\rho}$ . Therefore, the condition  $c'(\frac{\hat{p}}{\eta} - \theta) = V'(\hat{p})$  simplifies to  $c'(\frac{\hat{p}}{\eta} - \theta) = \frac{b'(\hat{p}) - \frac{1}{\eta}c'(\frac{\hat{p}}{\eta} - \theta)}{\rho}$ . This is equivalent to

$$X(\hat{p}) = b'(\hat{p}) \frac{1}{\rho + \frac{1}{\eta}} - c'(\frac{1}{\eta}\hat{p} - \theta) = 0 \quad (2)$$

Next, we see that given the conditions in the theorem there will be a unique solution to the above equation. If  $0 < b'(0) < \infty$  then  $X(0) > 0$  because  $c'(x)$  for  $x < 0$  is zero ( $c(x)$  is zero for all  $x < 0$ ) and since  $\lim_{p \rightarrow \infty} X(p) < 0$ . From the continuity of  $X(p)$  it follows that there exists a steady state  $\hat{p}$ . Observe that  $X(p)$  is decreasing in  $p$ , which follows from concavity of  $b(p)$  and  $-c(\frac{1}{\eta}p - \theta)$ . Hence, we can see that the steady state will be unique. This shows the first part of the theorem.

For the second part, we need to show that if  $p(0) < \hat{p}$  then the  $\lambda_{p(0)}(t)$  will decrease with time and settle to  $\hat{\lambda}$ , while the corresponding population  $p_{p(0)}(t)$  will increase and converge to  $\hat{p}$ .  $\frac{dp}{dt} = v(p) = \theta + c'^{-1}(V'(p)) - \frac{p}{\eta}$ . From the analysis presented above we know that  $v(p)$  is a decreasing function which starts at a positive value and then decreases to a negative value and intersects at the steady state value of the population. Note that if  $p(0) < \hat{p}$  then the  $v(p(0)) > 0$  is positive initially. Consider the population level  $v(\hat{p} - \epsilon)$  where  $\hat{p} - p(0) > \epsilon > 0$  then it can be seen easily that the population will grow as fast as  $\frac{dp}{dt} \geq v(\hat{p} - \epsilon)$ . Hence, this can be done for any  $\epsilon$ , which proves convergence.  $\lambda = c'^{-1}(V'(p))$ , note that  $V'(p)$  is a decreasing function in  $p$ . For this we need to show that  $V$  is concave, which follows from [2]. We also give the formal argument here.

Consider two initial population levels  $p_1, p_2$  and the corresponding optimal referral policy and population trajectories are given as  $(\lambda_1(t), \lambda_2(t))$  and  $(p_1(t), p_2(t))$  respectively. Let us consider convex combination of  $p_1$  and  $p_2$ ,  $p_3 = \kappa p_1 + (1 - \kappa)p_2$ . We know that the value function  $V(p_3) \geq \int_0^\infty e^{-\rho t} (b(p_3(t)) - c(\lambda_3(t)))$ , where  $p_3(t)$  is the population trajectory when  $\lambda_3(t) = \kappa \lambda_1(t) + (1 - \kappa)\lambda_2(t)$ . From the linearity of the differential equation for  $p$ , we can see that  $p_3(t)$  will be  $\kappa p_1(t) + (1 - \kappa)p_2(t)$ .  $V(p_3) \geq \int_0^\infty e^{-\rho t} (b(\kappa p_1(t) + (1 - \kappa)p_2(t)) - c(\lambda_3(t) = \kappa \lambda_1(t) + (1 - \kappa)\lambda_2(t)))$  and from the concavity of  $b(p)$  and  $-c(\lambda)$  we can say that  $V(p_3) \geq \kappa V(p_1) + (1 - \kappa)V(p_2)$ . Hence  $V$  is concave. Hence, as pop-

ulation increases  $\lambda$  will decrease with time and converge.

□

Theorem 2. Policy comparison with change in  $\eta$  in steady state  $t \rightarrow \infty$ : Local Behavior

i) If the average time that a user stays on the website  $\eta$  is decreased by  $\epsilon > 0$  and if  $\Phi_{\theta,\eta,\rho}(b(\hat{p}), \hat{p}) < 0$  then the advertisements in the steady state increase.

ii) If the average time that a user stays on the website  $\eta$  is decreased by  $\epsilon > 0$  and if  $\Phi_{\theta,\eta,\rho}(b(\hat{p}), \hat{p}) > 0$  then the advertisements in the steady state decrease.

PROOF. i) In steady state the value function can be computed as follows  $V(\hat{p}) = \frac{b(\hat{p}) - c(\hat{\lambda})}{\rho}$ , where  $\hat{\lambda} = \frac{1}{\eta}\hat{p} - \theta$ . Differentiating  $V(\hat{p})$  w.r.t  $\hat{p}$ , we get  $\rho V'(\hat{p}) = b'(\hat{p}) - c'(\hat{\lambda})\frac{1}{\eta}$  (See Proof of Theorem 1.). We also know that  $V'(p) = c'(\hat{\lambda})$ , substituting  $V'(\hat{p})$  we get  $c'(\hat{\lambda})(\rho + \frac{1}{\eta}) = b'(\hat{p})$ . Expressing this equation in terms of only  $\hat{\lambda}$ , we get  $b'((\theta + \hat{\lambda})\eta)\frac{1}{\rho + \frac{1}{\eta}} = c'(\hat{\lambda})$ . Note that  $g(\hat{\lambda}) = b'((\theta + \hat{\lambda})\eta)\frac{1}{\rho + \frac{1}{\eta}} - c'(\hat{\lambda})$  is a continuously differentiable decreasing function in  $\hat{\lambda}$ . We are interested in determining how the behavior of the optimal referrals change when the rate at which users leave  $\lambda_d = \frac{1}{\eta}$  is increased by a small amount. Note that we will use  $\lambda_d = \frac{1}{\eta}$  in the equations that follow. Let  $\frac{dg'(\hat{\lambda})}{d\lambda_d} = -\frac{1}{\rho + \lambda_d}(b''(\frac{\theta + \hat{\lambda}}{\lambda_d})\frac{(\theta + \hat{\lambda})}{\lambda_d^2} + b'(\frac{\theta + \hat{\lambda}}{\lambda_d})\frac{1}{(\rho + \lambda_d)})$ . Let us consider the case when  $b''(\frac{\theta + \hat{\lambda}}{\lambda_d})\frac{(\theta + \hat{\lambda})}{\lambda_d^2} + b'(\frac{\theta + \hat{\lambda}}{\lambda_d})\frac{1}{(\rho + \lambda_d)} < 0$ , which is equivalent to  $\Phi_{\theta,\eta,\rho}(b(\hat{p}), \hat{p}) = \frac{d^2b(\hat{p})}{dp^2}\hat{p}\eta + \frac{db(\hat{p})}{dp_s}\frac{1}{\rho + \frac{1}{\eta}} < 0$ . In

this case  $\frac{dg'(\hat{\lambda})}{d\lambda_d} > 0$  and we claim that the optimal  $\hat{\lambda}$  will increase. We know that changing  $\lambda_d$  will lead to a new value for  $\hat{\lambda}$ , which will be unique. Let us assume that  $\hat{\lambda}$  decreases, which would mean that  $c'(\hat{\lambda})$  will decrease as well. Also we know that  $\frac{dg'(\hat{\lambda})}{d\lambda_d} > 0$ , which means that  $g(\hat{\lambda})$  at the new value of  $\hat{\lambda}'$  will be positive. This violates the condition for optimality, hence  $\hat{\lambda}$  increases.

Note that in the above the critical thing to showing the result is  $\Phi_{\theta,\eta,\rho}(b(\hat{p}), \hat{p}) < 0$  and we only allowed for a small change in  $\lambda_d$  because we only have local information about the derivative at  $\hat{p}$ .

ii) Consider the other case  $\Phi_{\theta,\eta,\rho}(b(\hat{p}), \hat{p}) = \frac{d^2b(\hat{p})}{dp^2}\hat{p}\eta + \frac{db(\hat{p})}{dp_s}\frac{1}{\rho + \frac{1}{\eta}} > 0$ . In this case  $\frac{dg'(\hat{\lambda})}{d\lambda_d} < 0$ , which means that  $\hat{\lambda}$  cannot increase because this would make  $g(\hat{\lambda})$  negative. □

Result 1. Policy comparison with change in  $\eta$  in steady state  $t \rightarrow \infty$ : Global Behavior

If the average stay time of the users decreases upto  $\bar{\eta}$  then the firm should increase the intensity of advertisements in the steady state, while if the average stay time falls below  $\bar{\eta}$  then the firm should decrease the advertisements.

PROOF. For the above result we will consider benefit function to take the form,  $b(p) = p^a$ .  $\Phi_{\theta,\eta,\rho}(b(p))$  is given as  $(p^{a-1}\eta)((a-1) + \frac{1}{\rho\eta+1})$ . Observe that if  $\eta \geq \bar{\eta}$  then  $\Phi_{\theta,\eta,\rho}(b(p)) < 0$ , which leads to an increase in the policy. For the other part also we can see that if  $\eta \leq \bar{\eta}$  a decrease in the average stay time makes the firm increase the advertisements in the policy. □

Theorem 3. Policy comparison with change in  $\theta$  in steady state  $t \rightarrow \infty$ : Global Behavior

If the intensity of the direct arrivals  $\theta$  is increased then the intensity of advertisements in the steady state decreases.

PROOF. Note that in this proof as well, we would substitute  $\eta = \frac{1}{\lambda_d}$ . We know that in the steady state the following  $b'(\frac{\theta + \hat{\lambda}}{\lambda_d})\frac{1}{\rho + \lambda_d} = c'(\hat{\lambda})$  holds. If  $\theta$  is increased then the term on the left  $b'(\frac{\theta + \hat{\lambda}}{\lambda_d})\frac{1}{\rho + \lambda_d}$  decreases, this is due to the fact that  $b''(p) < 0$ . For the steady state equation to hold, i.e.  $b'(\frac{\theta + \hat{\lambda}}{\lambda_d})\frac{1}{\rho + \lambda_d} = c'(\hat{\lambda})$  the term on the right can only decrease, which happens only if  $\hat{\lambda}$  decreases. This combined with the fact that a solution  $\hat{\lambda}$  always exists and is unique, leads to the conclusion that  $\hat{\lambda}$  decreases. □

Theorem 4. Policy comparison with change in  $\eta$ : at finite time  $t$

i) If the average stay time of the user decreases to a value above the threshold  $\eta^{th}$  then the intensity of advertisements increases at all time instances  $t$  larger than some finite  $t^*$ .

ii) If the average stay time of the user  $\eta$  decreases to a value below the threshold  $\eta^{th}$  then the intensity of advertisements at every time instance  $t$  decreases.

PROOF. Next, we study the firm's behavior when the total capacity of the system is finite. Let us arrive at the analytical form for the referrals by solving the HJB equation for the case given as follows.

$$\rho V(p) = \max_{\lambda \geq 0} (-(\gamma - p)^2 - c\lambda^2 + V'(\lambda)(\theta + \lambda - \lambda_d p))$$

The solution for optimal  $\lambda$  is  $\lambda = \max\{\frac{1}{2c}V'(p), 0\}$ . It can be shown that the value function is concave. Therefore, we can see that  $\lambda = \max\{\frac{1}{2c}V'(p), 0\}$  is a decreasing function in  $p$ .

Since the optimal  $\lambda$  is different in different regimes we give a case by case analysis.

Assume that the rate at which users leave is high, i.e. a low average stay time  $\lambda_d \gamma \geq \theta$ . Recall that we are using  $\lambda_d = \frac{1}{\eta}$ . Let us assume that the value function  $V(p) = Ap^2 + 2Bp + C$  for  $p \geq 0$ . Let us consider the regime where  $V'(p) = 2Ap + 2B \geq 0$  and  $p \geq 0$ , which implies  $\lambda = \frac{1}{2c}(2Ap + 2B)$ . Substituting  $\lambda = \frac{1}{2c}(2Ap + 2B)$  and  $V(p) = Ap^2 + 2Bp + C$  in the HJB equation we get

$$\rho(Ap^2 + 2Bp + C) = (-1 - 2A\lambda_d + \frac{A^2}{c})p^2 + (2\gamma + \frac{AB}{c} + 2A\theta - B\lambda_d)p + B\theta + \frac{B^2}{4c}$$

In the above equation we used the fact that  $\lambda = \frac{Ap+B}{c} \geq 0$  and  $p \geq 0$ . If  $Ap + B \geq 0$  and  $p \geq 0$  for a measurable set of values  $p$  then the the solution to the above is found by equating the coefficients we get  $A = \frac{c(2\lambda_d + \rho) - \sqrt{c^2(2\lambda_d + \rho)^2 + 4c}}{2}$ ,  $B = \frac{A\theta + \gamma}{\lambda_d + \rho - \frac{A}{c}}$ . Substituting  $A$  and  $B$  in the expression  $Ap + B \geq 0$  and  $p \geq 0$ , we get the following threshold on  $p$ ,  $p^{th} = -\frac{B}{A} = \frac{(\lambda_d \gamma - \theta)(2\lambda_d c + \rho) + \sqrt{c(4 + 4c\lambda_d^2 + c\rho^2 + 4c\lambda_d \rho)}}{2(c\lambda_d^2 + c\lambda_d \rho + 1)} + \frac{(\gamma + (\lambda_d + \rho)c\theta)}{c\lambda_d^2 + c\lambda_d \rho + 1}$ . Note that  $p^{th} > 0$ . Hence, we know that there exists  $0 < p \leq p^{th}$ . For each  $0 < p \leq p^{th}$  the equation (4) has to hold, which is only possible if the coefficients on left of (4) are the same as

the coefficients on the right. Hence,  $A = \frac{c(2\lambda_d + \rho) - \sqrt{c^2(2\lambda_d + \rho)^2 + 4c}}{2}$ , and term decays and after a certain threshold  $t^*$  the effect of increase in the first term will dominate.  $\square$

$B = \frac{A\theta + \gamma}{\lambda_d + \rho - \frac{A}{c}}$ . Therefore, for  $0 < p \leq p^{th}$  the optimal policy will be  $\lambda = \frac{Ap + B}{c}$ . For  $p > p^{th}$  the referrals will have to be 0 because we know that referrals are decreasing function of  $p$  and they cannot go below zero. Having computed optimal referral policy is a function of the population level, we will now compute the optimal referral policy and the population levels as a function of time.

The optimal policy and the corresponding population dynamic, when the average stay time of the user is low, i.e.  $\frac{\gamma}{\theta} \geq \eta$  or  $\lambda_d \gamma \geq \theta$  and the initial population is low as well, i.e.  $p(0) < p^{th}$  is computed as follows.

$$\frac{dp}{dt} = \theta + \frac{(B + Ap)}{c} - \lambda_d p$$

The steady state of the above dynamic is  $\hat{p} = \frac{\theta + B}{\lambda_d - A} = \frac{\gamma + (\lambda_d + \rho)c\theta}{c\lambda_d^2 + c\lambda_d\rho + 1}$ . If  $p(0) \leq \hat{p}$  then  $\frac{dp}{dt}|_{p=p(0)}$  is positive. The population will increase and saturate to attain the steady state value  $\hat{p}$ . Since the population trajectory never crosses  $p^{th}$ ,  $\lambda = \frac{Ap + B}{c}$ . Solving the population dynamic above, we get

$$p(t) = \frac{(\gamma + (\lambda_d + \rho)c\theta)}{(c\lambda_d^2 + c\lambda_d\rho + 1)}(1 - e^{-\frac{\sqrt{c \cdot (c\rho^2 + 4c\lambda_d^2 + 4c\lambda_d\rho + 4) + c\rho}}{2c}t}) + p(0)e^{-\frac{\sqrt{c \cdot (c\rho^2 + 4c\lambda_d^2 + 4c\lambda_d\rho + 4) + c\rho}}{2c}t}$$

Since the population is increasing, the optimal referrals will decrease and saturate to attain  $\lambda = \lambda_d \hat{p} - \theta$ . Optimal referrals  $\lambda = Ap(t) + B$  are given as

$$\lambda(t) = \left( \frac{(\lambda_d \gamma - \theta)}{(c\lambda_d^2 + c\lambda_d\rho + 1)} + 2(-p(0) + \frac{e^{-\frac{\sqrt{c \cdot (c\rho^2 + 4c\lambda_d^2 + 4c\lambda_d\rho + 4) + c\rho}}{2c}t}}{2c\lambda_d + c\rho + \sqrt{c \cdot (c\rho^2 + 4c\lambda_d^2 + 4c\lambda_d\rho + 4)}}) \right)$$

If  $\eta \leq \eta^{th}$ , where  $\eta^{th} = \frac{1}{\max\{\frac{\theta + \sqrt{c^2\theta^2 + (\rho\theta + \gamma)c\gamma}}{\gamma}, \frac{1}{\sqrt{c}}\}}$  or equivalently  $\lambda_d \geq \frac{1}{\eta^{th}}$  and if  $\eta$  decreases then we need to argue that the optimal referrals will decrease at all times.

Observe that if  $\lambda_d \geq \frac{\theta + \sqrt{c^2\theta^2 + (\rho\theta + \gamma)c\gamma}}{\gamma}$  then the first term in  $\lambda(t)$ , i.e.  $\frac{(\lambda_d \gamma - \theta)}{c\lambda_d^2 + c\lambda_d\rho + 1}$  decreases with increase in  $\lambda_d$ . In addition if  $\lambda_d \geq \frac{1}{\sqrt{c}}$  then  $\frac{(\gamma + (\lambda_d + \rho)c\theta)}{(c\lambda_d^2 + c\lambda_d\rho + 1)}$  decreases as well. Also,  $\frac{e^{-\frac{\sqrt{c \cdot (c\rho^2 + 4c\lambda_d^2 + 4c\lambda_d\rho + 4) + c\rho}}{2c}t}}{2c\lambda_d + c\rho + \sqrt{c \cdot (c\rho^2 + 4c\lambda_d^2 + 4c\lambda_d\rho + 4)}}$  decreases with increase in  $\lambda_d$ . Note that  $p(0) \leq \hat{p}$  ensures that

$2(-p(0) + \frac{(\gamma + (\lambda_d + \rho)c\theta)}{(c\lambda_d^2 + c\lambda_d\rho + 1)}) \frac{e^{-\frac{\sqrt{c \cdot (c\rho^2 + 4c\lambda_d^2 + 4c\lambda_d\rho + 4) + c\rho}}{2c}t}}{2c\lambda_d + c\rho + \sqrt{c \cdot (c\rho^2 + 4c\lambda_d^2 + 4c\lambda_d\rho + 4)}}$  decreases and this is true for all times  $t \geq 0$ .

Consider the case when  $\eta \geq \eta^{th}$ , where  $\eta^{th} = \frac{1}{\frac{\theta + \sqrt{c^2\theta^2 + (\rho\theta + \gamma)c\gamma}}{\gamma}}$ , this is equivalent to  $\lambda_d \leq \frac{\theta + \sqrt{c^2\theta^2 + (\rho\theta + \gamma)c\gamma}}{\gamma}$ . Observe that under this condition the first term in  $\lambda(t)$ , i.e.  $\frac{(\lambda_d \gamma - \theta)}{c\lambda_d^2 + c\lambda_d\rho + 1}$  increases, while the second term can increase/decrease. However, note that as  $t$  increases the effect of the change in sec-

*Uncertain user arrivals/exits:* In this section we allow for uncertainty in the arrivals and exits of users. This modifies the user population differential equation to the following stochastic differential equation,  $dP_t = (\theta + \lambda(t) - \lambda_d P)dt + \sigma P dW_t$ . (Recall that  $\lambda_d = \frac{1}{\eta}$ ) The HJB equation corresponding to this case is

$$\rho V(p) = \max_{\lambda} (\gamma^2 - (\gamma - p)^2 - c\lambda^2 + V'(p)(\theta + \lambda - \lambda_d p) + \frac{1}{2}V''(p)\sigma^2)$$

In the above setting we allow for negative referrals for analytical tractability, but we do not allow that the population becomes negative. Hence, the optimal  $\lambda$  should ensure that the population is always positive. Solving the optimal  $\lambda$  we get  $\lambda = \frac{V'(p)}{2c}$ . Let's assume that  $V(p) = A'p^2 + B'p + C'$

and substitute  $\lambda = \frac{V'(p)}{2c} = \frac{2A'p + B'}{2c}$  in the above HJB equation. Equating coefficients on both the sides we get  $A' = \frac{2\lambda_d + \rho - \sqrt{(2\lambda_d + \rho)^2 + 4(\frac{1}{c} + \sigma^2)}}{2(\frac{1}{c} + \sigma^2)}$  and  $B' = 2 \frac{(A'\theta + \gamma)}{(\rho + \lambda_d - \frac{A'}{c})}$ . Note that

$\frac{A'}{c} - \lambda_d < 0$  and  $\theta + \frac{B'}{2c} > 0$ , these conditions together with the fact that the initial population  $p(0) \geq 0$  ensure that the population dynamic stays above zero (see [6]). The expected population at a time  $t$  is given as  $E[P(t)] = \frac{\theta + \frac{B'}{2c}}{\lambda_d - \frac{A'}{c}}(1 - e^{-(\lambda_d - \frac{A'}{c})t}) + p(0)e^{-(\lambda_d - \frac{A'}{c})t}$ . The expected number of referrals are given as  $E[\lambda(t)] = \frac{A'E[P(t)] + B'}{2c}$ .

If  $p(0) \geq \frac{\theta + \frac{B'}{2c}}{\lambda_d - \frac{A'}{c}} = \frac{(\lambda_d + \rho)\theta + \gamma}{(\lambda_d + \rho - \frac{A'}{c})(\lambda_d - \frac{A'}{c})}$  then the expected population  $E[P(t)]$  dynamic decreases with time and converges to attain  $\frac{\theta + \frac{B'}{2c}}{\lambda_d - \frac{A'}{c}}$ . The corresponding trajectory of expected number of referrals will increase with time, this is because the referrals decrease with increase in population.

If  $p(0) \leq \frac{\theta + \frac{B'}{2c}}{\lambda_d - \frac{A'}{c}}$  then the expected population  $E[P(t)]$  dynamic increases with time and converges to attain  $\frac{\theta + \frac{B'}{2c}}{\lambda_d - \frac{A'}{c}}$ . The corresponding trajectory of expected number of referrals will decrease with time, this is because the referrals decrease with increase in population

**THEOREM 5.** *The average number of referrals given to a population of size  $P$  increases with in an increase in the variance  $\sigma^2$  of the Brownian motion.. The average number of referrals given at any time  $t$  increases with in an increase in the variance  $\sigma^2$  of the Brownian motion provided the initial population of the users is sufficiently high.*

We know that  $\lambda = \frac{A'}{c}P + \frac{B'}{2c}$ . We know that  $B' = 2 \frac{A'\theta + \gamma}{\rho + \lambda_d - \frac{A'}{c}}$  and from the expression observe that if  $A'$  increases  $B'$  increases as well. Next, we show that as  $\sigma$  increases then  $A'$  increases.

$$\frac{dA'}{d\sigma} = \frac{\sigma((2\lambda_d + \rho)^2 + 2(\frac{1}{c})) + 2\sigma^3}{\sqrt{(2\lambda_d + \rho)^2 + 4(\frac{1}{c} + \sigma^2)}(\frac{1}{c} + \sigma^2)}$$

Since  $\frac{dA'}{d\sigma} > 0$ , it shows the first part of the theorem.

We know that  $A$  and  $B$  increase with increase with  $\sigma$ . In addition if we can show that  $E[P(t)]$  increases as well, then the  $E[\lambda(t)]$  will increase as well. The expression for  $E[P(t)]$  is sum of two terms, the first term  $\frac{\theta + \frac{B'}{2c}}{\lambda_d - \frac{A'}{c}}$  clearly increases with  $\sigma$ . The second term is given by  $(p(0) - \frac{\theta + \frac{B'}{2c}}{\lambda_d - \frac{A'}{c}})e^{-(\lambda_d - \frac{A'}{c})t}$ . If  $p(0) \geq \frac{\theta + \frac{B'}{2c}}{\lambda_d - \frac{A'}{c}} + \frac{\theta + \frac{B'}{2c}}{(\lambda_d - \frac{A'}{c})^2} + \frac{1}{\lambda_d - \frac{A'}{c}} \frac{\frac{dB}{d\sigma}}{\frac{dA}{d\sigma}}$  is satisfied then the second term decreases as well. This is a sufficient condition and it can be checked by replacing in the derivative of the second term.

Hence, the  $E[\lambda(t)]$  will increase.