Robust Equilibria in Additively Coupled Games in Communications Networks

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***Abstract*—We obtain the robust Nash equilibrium (RNE) for a wide range of multi-user communications networks under uncertainty by utilizing the robust optimization theory for the worst-case uncertainties. To do so, we consider the uncertainty as a distance between the estimated and the actual values of the system parameters as a general norm function, and utilize the finite-dimensions variational inequalities (*VI*) to derive the conditions for existence and uniqueness of RNE. Two effects of uncertainty on the performance of the system are investigated: the difference between the achieved social utility at the RNE and the Nash equilibrium (NE) of the nominal game, and the distance between the deployed strategies of users at the RNE and at the NE. We quantify these two effects for the cases of unique NE and multiple NEs, and show that when the NE is unique, the achieved social utility at the RNE is always less than that of the NE. Interestingly, the worst-case robustness approach may lead to a higher social utility at the RNE in the multiple NEs scenario. Considering uncertainty at RNE introduces coupling between users, and hence, developing distributed algorithms for reaching RNE is more challenging as compared to the NE in the nominal game. However, for some special forms of utilities and norm functions, we propose simultaneous and sequential distributed algorithms; and investigate the performance of the robust game for power control in interference channels, and for flow control in Jackson networks.**

***Index Terms*—Robust game theory, sensitivity analysis, uncertainty region, variational inequality.**

I. Introduction

During the past decade, game theory has been widely used to analyze the performance of multi-user communication networks. In this context, transmitter-receiver pairs are considered as rational and self-interested players that aim to maximize their own utility by choosing their transmission strategy. To analyze the equilibrium of such a system, the notion of Nash equilibrium (NE) is frequently used, at which no user can attain a higher utility by unilaterally changing its strategy [1].

In addition, deriving the conditions for uniqueness and Pareto optimality of the NE are critical for any communications network analyzed via game theory. Such issues are investigated in [2]–[6] for different system models of flow and congestion control, network routing, and power control problems. Besides, in [7] many such games in communications networks are considered that exhibit some unique features such as: 1) each user has a multi-dimensional strategy subject to a single sum resource constraint; 2) each user's utility in each dimension is impacted by an additive combination of its own action and other users' actions; 3) users' utilities are separable across different dimensions; and 4) each user's utility is obtained by summation over all dimensions.

Undoubtedly, there are many sources of uncertainties in such networks that emanate from, inter alia, lack of coordination between users, asynchronous transmissions of peers in the network, and variations in system parameters. Hence, determining the effect of uncertainties in such environments and proposing an appropriate approach to overcome their undesirable effects are important and under-explored areas of research. To tackle uncertainty, there are two distinct approaches in the robust optimization theory: the Bayesian (or probabilistic) approach, where the statistics of uncertainties are considered and the performance is stochastically guaranteed; and the worst-case-uncertainty approach, where the uncertain parameters are modeled as deterministic values bounded in a closed region (called the uncertainty region) and the performance is guaranteed for any realization of uncertainty within that region [8]. In many deployment scenarios, the worst-case-uncertainty approach is more practical to characterize the network's performance [9], [10].

Recently, for the problem of allocating transmit power levles in interference channels, the Bayesian and the worst-case uncertainty approaches are proposed by [9], [11]–[13], where the uncertain parameters are the channel gains between different transmitter and receiver pairs. In this paper, we use the worst-case uncertainty approach to introduce a robust equilibrium for a general class of games defined in [7]. Our main objective in this paper is to present a complete analysis of robust Nash equilibrium (RNE) in robust games as compared to that of NE in nominal games with complete information. In doing so, we endeavor to answer the following basic questions that are pertinent to robust games: 1) How should we express the uncertainty region so that the RNE is tenable to analysis and can be shown to always exist? 2) What is the condition for uniqueness of the RNE in a robust game for a given uncertainty region? 3) How are the RNE and the NE related, and what is the impact of robustness on the social utility and on users' strategies at the RNE as compared to the NE? 4) Is there a distributed algorithm for reaching the RNE, and what are the conditions for its convergence? We will show that the uniqueness condition for the RNE and the NE can be unified by using variational inequalities (*VI*), obtain the upper bound of the difference between strategies of users at the RNE and at the NE, and demonstrate that when the RNE and the NE are unique, the social utility at the RNE is less than that of the NE.

The rest of this paper is organized as follows. In Section II, we present the system model. In Section III, we introduce the notion of robust games and the RNE via the worst-case uncertainty approach, and obtain the conditions for the RNE's existence. Section IV covers the conditions for RNE's uniqueness, followed by Section V, where for the case of linear interactions between users, the RNE is obtained via affine variational inequalities (*AVI*), and distributed algorithms for reaching the RNE are proposed. In Section VI, we provide simulation results to validate our analysis for the power allocation problem and for Jackson networks. Finally, conclusions are drawn in Section VII.

II. System Model

Consider a strategic-form game $\mathcal{G} = \{\mathcal{N}, (v_n(\mathbf{a}))_{n\in\mathcal{N}}, \mathcal{A}\}$, where $\mathcal{N} = \{1, \cdots, N\}$ denotes the set of selfish users (i.e., players of the game), $\mathcal{A} = \prod_{n\in\mathcal{N}} \mathcal{A}_n$ is the joint strategy space of the game, $\mathcal{A}_n \subseteq \mathbb{R}^K$ is the strategy space of user n, $v_n(\mathbf{a}) : \mathcal{A} \to \mathbb{R}$ is the n^{th} user's utility function that depends on the chosen strategy vector $\mathbf{a} = [\mathbf{a}_n, \mathbf{a}_{-n}]$ of all users, $\mathbf{a}_n \in \mathcal{A}_n$ is the action of user n, $\mathbf{a}_{-n} \in \mathcal{A}_{-n}$ denotes the actions of all users except user n, and $\mathcal{A}_{-n} = \prod_{m\in\mathcal{N}, m\neq n} \mathcal{A}_m$ is the strategy space of all users except user n. In a non-cooperative strategic game, each user aims to maximize its own utility as

$$\max_{\mathbf{a}_n\in\mathcal{A}_n} v_n(\mathbf{a}_n, \mathbf{a}_{-n}), \quad \forall n \in \mathcal{N}. \tag{1}$$

Interactions between users are studied using the notion of NE, which corresponds to the strategy profile $\mathbf{a}^* = (\mathbf{a}_1^*, \cdots, \mathbf{a}_N^*)$ such that for any other strategy profile, we have $v_n(\mathbf{a}_n^*, \mathbf{a}_{-n}^*) \geq v_n(\mathbf{a}_n, \mathbf{a}_{-n}^*)$ for all $n \in \mathcal{N}$.

***Definition* 1.** The game $\mathcal{G}$ is said to be additively coupled (ACG) [7] when

1. For all user in $\mathcal{N}$, the strategy space is

$$\mathcal{A}_n = \{\mathbf{a}_n = (a_n^1, \cdots, a_n^K) | a_n^k \in [a_{n,k}^{\min}, a_{n,k}^{\max}] \sum_{k=1}^K a_n^k \leq a_n^{\max}\} \tag{2}$$

2. The utility function of each user is $v_n(\mathbf{a}_n, \mathbf{a}_{-n}) = \sum_{k=1}^K [y_n^k(a_n^k + f_n^k(\mathbf{a}_{-n})) - g_n^k(\mathbf{a}_{-n})]$, where $y_n^k : \mathbb{R} \to \mathbb{R}$ is an increasing, twice differentiable, and strictly concave function in $\mathbf{a}_n$, which represents the direct effect of each user's strategy on its utility function; and $f_n^k(\mathbf{a}_{-n}) : \mathbb{R}^{(N-1)K} \to \mathbb{R}$ and $g_n^k(\mathbf{a}_{-n}) : \mathbb{R}^{(N-1)K} \to \mathbb{R}$ are both twice differentiable functions of system parameters, and denote the effect of other users' strategies on the utility of user n. Different examples of this type of game are introduced by [7].

Since $v_n(\mathbf{a}_n, \mathbf{a}_{-n})$ is a continuous function in $\mathbf{a}$, which is quasi-concave for $\mathbf{a}_n \in \mathcal{A}_n$, and the strategy space of users is convex and compact, it can be proved that a NE always exists [1]. The conditions for uniqueness of NE and a distributed algorithm for reaching NE can be obtained by using the best-response algorithms and principles of contraction mapping [5]–[7], (see [1] for more detail).

III. Robust Games

As stated earlier, users may encounter different sources of uncertainty caused by variations in $\mathbf{a}_{-n}$ and/or in system parameters, which cause variations in the utility of each user, and prevent users from attaining their expected performances. To tackle such issues, we assume that all uncertainties for a given user can be modeled by variations in f_n^k, i.e., $\widetilde{f}_n^k(\mathbf{a}_{-n}) = f_n^k(\mathbf{a}_{-n}) + \hat{f}_n^k(\mathbf{a}_{-n})$, where $\widetilde{f}_n^k(\mathbf{a}_{-n})$ is the actual value, $f_n^k(\mathbf{a}_{-n})$ is the nominal value estimated by user n, which is the same as in the nominal game, and $\hat{f}_n^k(\mathbf{a}_{-n})$ is the error due to uncertainty for user n, which may not be the same for all users. In what follows, we omit $\mathbf{a}_{-n}$ from $\widetilde{f}_n^k$ and $\hat{f}_n^k$ for simplicity in notations.

In robust optimization for the worst-case uncertainty, error variations are assumed to be bounded, hence the actual values of uncertain parameters are always bounded in an uncertainty region defined as [14]

$$\Re_n = \{\widetilde{\mathbf{f}}_n \in \mathbb{R}^K | \, \|\hat{\mathbf{f}}_n\| \leq \epsilon_n\}, \quad \forall n \in \mathcal{N} \tag{3}$$

where $\widetilde{\mathbf{f}}_n = [\widetilde{f}_n^1, \cdots, \widetilde{f}_n^k]$, $\mathbf{f}_n = [f_n^1, \cdots, f_n^k]$, and $\hat{\mathbf{f}}_n = [\hat{f}_n^1, \cdots, \hat{f}_n^k]$; and $\|\cdot\|$ denotes any definition of norm. Since norm is a convex function, the uncertainty region described by (3) is a closed and convex set [15].

For different sources of uncertainty, different forms of uncertainty regions are introduced in [10], [16]. For example, uncertainty in the channel gain caused by estimation errors due to Gaussian noise can be modeled by a spherical region centered at the nominal (estimated) channel gain, whose radius is a function of the noise power and its distribution, i.e., the weighted norm with $p = 2$.

The effect of uncertainty in $\widetilde{\mathbf{f}}_n$ is highlighted by a new variable in the utility function of each user denoted by $u_n(\mathbf{a}_n, \mathbf{a}_{-n}, \widetilde{\mathbf{f}}_n) = \sum_{k=1}^K [y_n^k(a_n^k + \widetilde{f}_n^k) - g_n^k(\mathbf{a}_{-n})]$. The objective of the worst-case-uncertainty approach is to find the optimal strategy for each user that optimizes each user's utility under the worst performance for any error in the uncertainty region. Therefore, the utility function in the worst-case-uncertainty approach can be formulated [17] as

$$\widetilde{u}_n = \max_{\mathbf{a}_n\in\mathcal{A}_n} \min_{\widetilde{\mathbf{f}}_n\in\Re_n} u_n(\mathbf{a}_n, \mathbf{a}_{-n}, \widetilde{\mathbf{f}}_n) \tag{4}$$

We refer to the corresponding game whose utility function is (4) as the robust additively coupled game (RACG), denoted by $\widetilde{\mathcal{G}} = \{\mathcal{N}, (\widetilde{u}_n)_{n\in\mathcal{N}}, \mathcal{A}\}$. Note that the strategy space of users in RACG is the same as that in ACG, but the solution to (4) for user n is a pair $(\widetilde{\mathbf{a}}_n', \widetilde{\mathbf{f}}_n') \in \mathcal{A}_n \times \Re_n$ that satisfies [18]

$$\max_{\mathbf{a}_n\in\mathcal{A}_n} u_n(\mathbf{a}_n, \mathbf{a}_{-n}, \widetilde{\mathbf{f}}_n') = u_n(\widetilde{\mathbf{a}}_n', \mathbf{a}_{-n}, \widetilde{\mathbf{f}}_n') = \min_{\widetilde{\mathbf{f}}_n\in\Re_n} u_n(\widetilde{\mathbf{a}}_n', \mathbf{a}_{-n}, \widetilde{\mathbf{f}}_n)$$

which is the saddle point of (4). Using the above, the equilibrium of the robust game $\widetilde{\mathcal{G}}$, i.e., the RNE is defined below.

The RNE of the RACG corresponds to the strategy profile $\widetilde{\mathbf{a}}^* = (\widetilde{\mathbf{a}}_1^*, \cdots, \widetilde{\mathbf{a}}_N^*)$ if and only if for any other strategy profile $\widetilde{\mathbf{a}}_n$ we have [17]

$$\min_{\widetilde{\mathbf{f}}_n \in \Re_n} u_n(\widetilde{\mathbf{a}}_n^*, \widetilde{\mathbf{a}}_{-n}^*, \widetilde{\mathbf{f}}_n) \geq \min_{\widetilde{\mathbf{f}}_n \in \Re_n} u_n(\widetilde{\mathbf{a}}_n, \widetilde{\mathbf{a}}_{-n}^*, \widetilde{\mathbf{f}}_n), \quad \forall\, \widetilde{\mathbf{a}}_n \in \mathcal{A}_n \tag{5}$$

***Lemma* 1**: When the uncertainty region for the system parameters is as in (3), the RNE's existence in the RACG is guaranteed.

Proof: See Appendix A. ■

IV. RNE's Uniqueness Conditions

For the RNE, the best-response of the utility function for each user cannot be derived in a closed-form, hence obtaining the uniqueness conditions via the fixed-point algorithm and contraction mapping is not possible [5], [7]. To resolve this, we model the ACG's NE using the notion of variational inequality (*VI*), and derive the NE's uniqness conditions. We also study the RACG's RNE using sensitivity analysis of *VI*, and obtain its characteristics. Recall that the utility of user n is v_n and u_n for the nominal and the robust games, respectively.

A. ACG's NE Analysis

The ACG's NE can be obtained by solving $VI(\mathcal{A}, \mathcal{F})$, where $\mathcal{F}_n = (-\frac{\partial v(\mathbf{a}_n, \mathbf{a}_{-n})}{\partial \mathbf{a}_n})$, $\mathcal{F} = (\mathcal{F}_n)_{i=1}^N$ (Proposition 1.4.2 in [18]). Hence, the conditions for RNE's uniqueness can be obtained from monotonicity in mapping $\mathcal{F}$ (Propositions 12.9 and 12.12 in [19]). Consider the following definitions of mapping $\mathcal{F}$

$$\begin{aligned} \alpha_n(\mathbf{a}) &\triangleq \text{smallest eigenvalue of} \quad -\nabla^2_{\mathbf{a}_n} v_n \\ &\Longrightarrow \alpha_n^{\min} \triangleq \inf_{\mathbf{a} \in \mathcal{A}} \alpha_n(\mathbf{a}) \quad \forall n \in \mathcal{N} \end{aligned} \tag{6}$$

$$\begin{aligned} \beta_{nm}(\mathbf{a}) &\triangleq \|\nabla_{\mathbf{a}_n \mathbf{a}_m} v_n\| \quad \forall n \neq m \\ &\Longrightarrow \beta_{nm}^{\max} \triangleq \sup_{\mathbf{a} \in \mathcal{A}} \beta_n(\mathbf{a}) \quad \forall n \in \mathcal{N} \end{aligned} \tag{7}$$

From the above, we define the $N \times N$ matrix

$$[\Upsilon]_{nm} = \begin{cases} \alpha_n^{\min} & \text{if} \quad m = n \\ -\beta_{nm}^{\max} & \text{if} \quad m \neq n \end{cases}$$

When $\boldsymbol{\Upsilon}$ is a P-matrix, the mapping $\mathcal{F}$ is strictly monotone and the NE is unique (Theorem 12.5 in [19]).

B. RACG's RNE Analysis

When the system encounters uncertainty, the actual values $\widetilde{f}_n^k$ are used instead of the nominal values f_n^k in the mapping $\mathcal{F}$. So considering uncertainty in interactions between users as defined in (3) can be viewed as perturbations in the mapping $\mathcal{F}$ of the nominal game. As such, the RNE in $\widetilde{\mathcal{G}}$ is equivalent to the perturbed solution of $VI(\mathcal{A}, \mathcal{F})$, denoted by $VI(\mathcal{A}, \widetilde{\mathcal{F}})$, where $\widetilde{\mathcal{F}}$ is the perturbed $\mathcal{F}$ with the value $\widetilde{f}_n^k = f_n^k + \hat{f}_n^k$.

***Theorem* 1**: For $\widetilde{\mathcal{G}}$, when $\boldsymbol{\Upsilon}$ is a P-matrix, for any value of $\boldsymbol{\Delta} = [\epsilon_1, \cdots, \epsilon_N]$, we have: 1) both RNE and NE are unique; 2) the social utility at RNE is always less than that at NE; and 3) the distance between the strategy spaces at RNE and at NE is

$$\|\mathbf{a}^* - \widetilde{\mathbf{a}}^*\|_2 \leq \min\{\frac{\|\boldsymbol{\Delta}\|_2^2}{c_{\text{sm}}(\mathcal{F})}\} \tag{8}$$

where c_{sm} is the strong monotonicity constant for mapping $\mathcal{F}$.

Proof: See Appendix B. ■

From Theorem 1, the performance of RACG can be examined and compared to that of ACG through the difference between the social utility at RNE and at NE, and their upper bound variations using (8). Note that $\mathbf{a}_n^*$ is the attractor for all solutions $VI(\mathcal{A}, \widetilde{\mathcal{F}})$ (Proposition 2.4.10 in [18]), i.e., $\lim_{\boldsymbol{\Delta} \to 0} \sup\{\|\mathbf{a}^* - \widetilde{\mathbf{a}}^*\|_2 : \mathbf{a}^*, \widetilde{\mathbf{a}}^* \in \mathcal{A}\} = 0$, meaning that when uncertainty approaches zero, RNE converges to NE (see the illustrative example for Theorem 1 in Section IV of [20]). Besides, when the uncertainty region is closed and convex, the RNE's uniqueness condition is the same as of the NE. In general, for multiple NE cases, one cannot analytically determine whether introducing robustness against uncertainty would increase or decrease the social utility at RNE, because social utility is a non-smooth and non-convex function with many local optima. As such, depending on the amount of uncertainty, the equilibrium may switch from one local optima to another.

V. Linear Interactions between Users

Let f_n^k be a linear function of other users' strategies, i.e.,

$$f_n^k = \sum_{m \neq n} F_{mn}^k a_m^k, \tag{9}$$

where $F_{mn}^k \in \mathbb{R}$ is a real value that depends on the system parameters. Moreover, let the utility of each user be [21]

$$y_n^k = \begin{cases} \log(\alpha_n^k + F_{nn}^k a_n^k), & \text{if} \quad \theta = 1 \\ \frac{(\alpha_n^k + F_{nn}^k a_n^k)^{\theta+1}}{\theta+1} & \text{if} \quad -1 < \theta < 0 \quad \text{or} \quad \theta < -1 \end{cases} \tag{10}$$

We model the uncertain parameter by $\widetilde{F}_{mn}^k = F_{mn}^k + \widehat{F}_{mn}^k$, and describe the uncertainty region by

$$\Re_n^k = \{\widetilde{\mathbf{F}}_{mn} | \, , \|\widehat{\mathbf{F}}_{mn}\| \leq \epsilon_n^k\}, \qquad \forall k \tag{11}$$

where $\widetilde{\mathbf{F}}_{mn} = [\widetilde{F}_{mn}^1, \cdots, \widetilde{F}_{mn}^K]$, $\widehat{\mathbf{F}}_{mn} = [\widehat{F}_{mn}^1, \cdots, \widehat{F}_{mn}^K]$, and ϵ_n^k is the bound on the error for each k.

***Proposition* 1**: When utility function is (10) and interactions between users are linear, we have

1) The game $\widetilde{\mathcal{G}}$ converges to its unique equilibrium if $\|\mathbf{M}^{\max} + \boldsymbol{E}\|_2 < 1$ where

$$[M^{\max}]_{mn} = \begin{cases} \max_{k \in [1, \cdots, K]} \frac{F_{mn}^k}{(F_{nn}^k)^{1+\frac{1}{\theta}}} & \text{if} \quad m \neq n \\ 0 & \text{otherwise} \end{cases}$$

$$[E]_{mn} = \begin{cases} \|\boldsymbol{\epsilon}_n\|_\infty & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

where $\boldsymbol{\epsilon}_n = [\epsilon_n^1, \cdots, \epsilon_n^K]$.

2) The total utility at RNE is always less than that at NE, and the upper bound of the strategy space of each users is $\|\mathbf{a}^* - \tilde{\mathbf{a}}^*\| \leq \frac{\|\boldsymbol{E}\|_2^2}{\lambda_{\min}(\mathbf{M}^{\max})}$, when $\mathbf{M}$ is a positive definite matrix.

Proof: See Appendix C. ■

The optimal solution of this type of game is

$$\widetilde{l}_n^k = [(\frac{1}{F_{nn}^k})^{1+\frac{1}{\theta}} \lambda_n^{\frac{1}{\theta}} - \frac{\alpha_n^k}{F_{nn}^k} - \sum_{m \neq n} F_{mn}^k a_m^k + \sum_{m \neq n} \max_{\widetilde{\mathbf{F}}_{mn} \in \Re_n^k} (\widetilde{F}_{mn}^k - F_{mn}^k) a_m^k]_{a_{nk}^{\min}}^{a_{nk}^{\max}}, \quad (12)$$

where the Lagrange multiplier λ_n for each user is chosen in such a way to satisfy the sum constraint $\sum_{k=1}^{K} a_n^k = a_n^{\max}$. The last part of (12) can be considered as the dual of linear norm [22]; hence, for linear norm with order p, it changes to linear norm with order $q = 1 + \frac{1}{p-1}$ [14].

The distributed and iterative algorithms for this type of utility functions and linear norms are obtained by using (12). In this context, two frequently used iterative algorithms are sequential algorithms, where users update their strategies sequentially according to a given schedule; and synchronous algorithms, where all users update their strategies at the same time, as shown in Table I. If Proposition 1.a holds, the best-response solution for (12) is a block-contraction mapping, meaning that from any arbitrary feasible point, the distributed algorithms converge to the unique fixed point (Proposition 1.1, Chapter 3 in [23]).

TABLE I
DISTRIBUTED ALGORITHMS

Algorithm 1: Synchronous Distributed Algorithm
For $t = 0$, set any feasible power allocation $\widetilde{\mathbf{a}}_n(0)$ for all $n \in \mathcal{N}$.
For $t = 1, \cdots, T$:
Calculate $\widetilde{\mathbf{a}}_n(t)$ from (12) $\forall n \in \mathcal{N}$,
and send the updated value of $\widetilde{\mathbf{a}}_n(t)$ to other users.
End.

Algorithm 2: Sequential Distributed Algorithm
For $t = 0$, set any feasible power allocation $\widetilde{\mathbf{a}}_n(0)$ for all $n \in \mathcal{N}$.
For $t = 1, \cdots, T$: $\forall n \in \mathcal{N}$, consider $w = \mathrm{mod}(t, N)$,
if $n = w$, calculate $\widetilde{\mathbf{a}}_n(t)$ from (12), $\forall n \in \mathcal{N}$,
and send updated value $\widetilde{\mathbf{a}}_n(t)$ to other users
Otherwise, $\widetilde{\mathbf{a}}_n(t+1) = \widetilde{\mathbf{a}}_n(t)$.
End.

VI. SIMULATION RESULTS

We use simulations to study the impact of robustness in power control in wireless networks, and also in Jackson networks, and to get an insight into the performance of $\widetilde{\mathcal{G}}$ for different bounds on uncertainty as compared to that of $\mathcal{G}$. In the following simulations, uncertainty for all users denoted by ϵ is assumed to be the same and is normalized, and each uncertainty region is modeled by a linear norm with order 2, i.e., an ellipsoid.

A. Power Control

For power control, we begin by studying the effect of uncertainty on the performance of both robust and non-robust approaches in terms of utility variations at their equilibriums. To do so, we consider $N = 2$ users and $K = 6$, and the amount of uncertainty is assumed to be $\epsilon = 50\%$ at the RNE. After convergence to the RNE and to the NE, the system parameter varies uniformly from 50% to 150%, which causes variations in the utility of each user at the NE and at the RNE, as shown in Fig. 1. Note that the total system utility varies considerably at the NE of the nominal game for both $\epsilon = 50\%$ and 150%, meaning that communication is very unreliable from the user's perspective. In contrast, the total system utility at the RNE of the robust game is stable in both cases. Although we assumed $\epsilon = 50\%$ in RACG, reliable transmissions is provided even at values higher than $\epsilon = 50\%$, e.g., up to 150%.

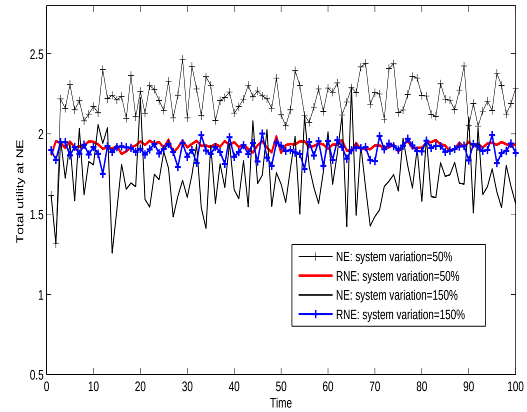

Fig. 1. The impact of channel variations in robust and non-robust games.

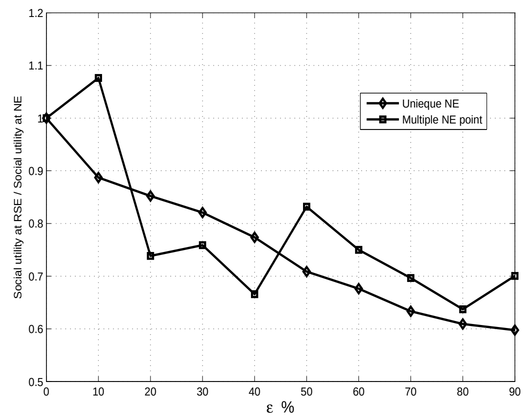

Fig. 2. Comparing RNE and NE for different uncertainty regions for unique NE and multiple NEs.

Next, in Fig. 2, we compare the effect of uncertainty when Proposition 1 holds, with that of the case when it does not hold, in terms of the ratio of the total achieved utility at the RNE and at the NE for different amounts of uncertainty. Simulations are performed for $N = 4$ and $K = 16$ in Rayleigh fading channels for bounded and uniformly generated errors for each cross sub-channel gain between two users. The ratio of the total achieved utility in Fig. 2 is obtained by averaging over 100 channel realizations. Note that when NE is unique, the total throughput of the system gradually decreases, but for the case of multiple NEs, no uniformity is observed.

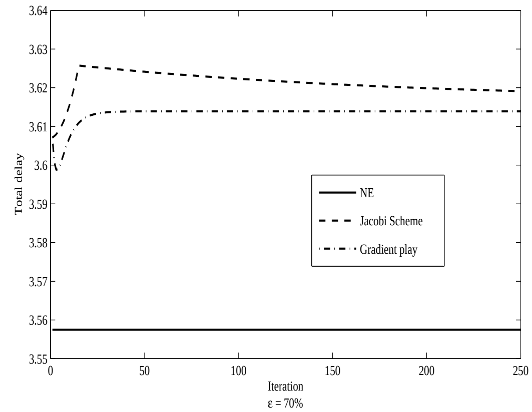

Fig. 3. The effect of uncertainty on the total delay of Jackson networks when $\boldsymbol{\epsilon} = 70\%$.

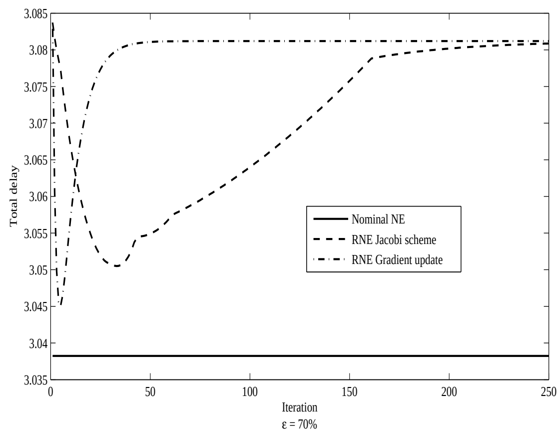

Fig. 4. Convergence to RNE for Jackson networks when $\boldsymbol{\epsilon} = 70\%$ for Jacobi scheme and gradient play approach.

B. Jackson Networks: M/M/1

To show the effect of uncertainty on the system performance in Jackson networks, we consider a network with $N = 5$ nodes and $K = 3$ traffic classes. Fig 3 shows the effect of uncertainty in transmission rates of users on the convergence of Jacobi scheme and the gradient approach for reaching NE. Without considering uncertainty, neither of the two algorithms converge to NE. Fig. 4 shows RNE for $\boldsymbol{\epsilon} = 70\%$. Note that in both cases, RNEs converge to the vicinity of the nominal NE, and that RNEs are stable.

VII. Conclusion

We showed that the notion RNE can be used in a wide range of problems in communication networks, in which each entity (user/player/node) selfishly competes with others to maximize its own utility by optimally selecting its bounded multidimensional strategy, where the system information is uncertain. In doing so, we considered uncertainty in the system's parameters by additive errors bounded in closed and convex regions, and applied the robust worst-case-uncertainty approach to the optimization problem of each entity. We also showed that the theory of finite-dimension *VI* and its sensitivity analysis can be used to obtain the conditions for the existence and uniqueness of the robust equilibria.

Appendix A
Proof of Lemma 1

As stated earlier, RNE is the saddle point of (4) for all users, and can be obtained by solving $VI(\widehat{\mathcal{A}}, \widehat{\mathcal{F}})$ [18], where $\widehat{\mathcal{F}} = (\widehat{\mathcal{F}}_n)_{n=1}^N$, and

$$\widehat{\mathcal{F}}_n \equiv \begin{pmatrix} -\frac{\partial u_n(\mathbf{a_n}, \mathbf{a_{-n}}, \widetilde{\mathbf{f}}_\mathbf{n})}{\partial \mathbf{a}_n} \\ \frac{\partial u_n(\mathbf{a_n}, \mathbf{a_{-n}}, \widetilde{\mathbf{f}}_\mathbf{n})}{\partial \widetilde{\mathbf{f}}_n} \end{pmatrix}, \tag{13}$$

where $\widehat{\mathcal{A}} = \prod_{n \in \mathcal{N}} \widehat{\mathcal{A}}_n$, and $\widehat{\mathcal{A}}_n = \mathcal{A}_n \times \Re_n$. Since y_n and f_n are twice differentiable functions, $\widehat{\mathcal{F}}$ is a continuous mapping. Also, the strategy space $\widehat{\mathcal{A}}_n$ is closed and convex because $\mathcal{A}_n$ and $\Re_f$ are closed and convex sets for all users. Therefore $VI(\widehat{\mathcal{A}}, \widehat{\mathcal{F}})$ always obtains the solution (Corollary 2.2.10 in [18]). Hence, RNE always exists for RACGs.

Appendix B
Proof of Theorem 1

1) Consider the bounded perturbation of mapping $\mathcal{F}(\mathbf{a})$ and $\widetilde{\mathcal{F}}(\mathbf{a})$ caused by variations in system parameters as $\mathcal{Q} = \|\mathcal{F}(\mathbf{a}) - \widetilde{\mathcal{F}}(\mathbf{a})\|_2 \quad \forall \mathbf{a}_n \in \mathcal{A}$. Since the strategy space of all users in each dimension is bounded as in (2), and the uncertainty region is bounded and convex, this region is also bounded, i.e., $q^{\max} = \max_{\mathbf{a} \in \mathcal{A}} \min_{\widetilde{\mathbf{f}}_n \in \Re_n} \|\mathcal{F}(\mathbf{a}) - \widetilde{\mathcal{F}}(\mathbf{a})\|_2 \leq \infty, \quad \forall n \in \mathcal{N}$. Any solution to the robust optimization for worst-case-uncertainty in (4) corresponds to a realization of $VI(\mathcal{A}, \widetilde{\mathcal{F}}) = VI(\mathcal{A}, \mathcal{F} + \mathbf{q})$, where $\mathbf{q} = q \times (\mathbf{1}_K^{\mathrm{T}})_1^N$ and $q \in \mathcal{Q}$ and $q \leq q^{\max}$. When $\mathcal{F}(\mathbf{a})$ is continuous and strictly monotone on the closed convex set $\mathcal{A}$, meaning that $\boldsymbol{\Upsilon}$ is a P matrix, the solution to $VI(\mathcal{A}, \mathcal{F} + \mathbf{q})$, denoted by $\Phi(\mathbf{q})$, is a monotone and single-valued mapping on its domain (Excercise 2.9.17 in [18]), i.e.,

$$\forall \mathbf{q}_i = q_i \times (\mathbf{1}_K^{\mathrm{T}})_1^N, \mathbf{q}_j = q_j \times (\mathbf{1}_K^{\mathrm{T}})_1^N, \; q_i, q_j \in \mathcal{Q} \\ \Longrightarrow (\Phi(\mathbf{q}_i) - \Phi(\mathbf{q}_j))(\mathbf{q}_i - \mathbf{q}_j) = 0, \tag{14}$$

Thus, when $q_i \neq q_j$, we have $\Phi(\mathbf{q}_i) = \Phi(\mathbf{q}_j)$, which is single valued on $\mathcal{Q}$, meaning a unique solution for all $q \in Q$. This completes the proof of the uniqueness of RNE under the P property of $\boldsymbol{\Upsilon}$.

2) Recall that when $\boldsymbol{\Upsilon}$ is a P matrix, $\mathcal{F}(\mathbf{a})$ is strictly monotone, and the utility is strictly convex. Since $\mathcal{A}$ is convex in $\mathbb{R}^K$, and $\mathcal{F}(\mathbf{a}) : K \to \mathbb{R}^K$ is a continuous mapping on $\mathcal{A}$, the solution to $VI(\mathcal{A}, \mathcal{F} + \mathbf{q})$ is always a compact and convex set (Corollary 2.6.4 in [18]). Also, since $\mathbf{a}_n^*$ is the optimum value of this convex set for $VI(\mathcal{A}, \mathcal{F})$, i.e., $q = 0$, any point in this set is less than $\mathbf{a}_n^*$, which is the solution to $VI(\mathcal{A}, \mathcal{F} + \mathbf{q})$. Note that $\widetilde{\mathbf{a}}_n^*$ belongs to this set. Since $\boldsymbol{\Upsilon}$ is a P matrix and $\mathcal{F}$ is strictly monotone, we have

$$\forall \; \mathbf{a}_n \leq \mathbf{a}_n^* \Longrightarrow u_n(\mathbf{a}_n, \mathbf{a}_{-n}) \leq u_n(\mathbf{a}_n^*, \mathbf{a}_{-n}^*) \quad \forall \mathbf{a} \in \mathcal{A} \tag{15}$$

which is also valid for $\widetilde{\mathbf{a}}_n^*$. As such, the utility at the RNE is less than that at the NE.

3) Since $\mathcal{F}$ is strongly monotone, there is a unique solution denoted by $\widetilde{\mathbf{a}}^* = \Phi^*(\mathbf{q})$, which can be considered as the robust solution for $\widetilde{\mathcal{G}}$ in the worst-case-uncertainty when $\|\mathbf{q}\|_2 \leq \|\mathbf{\Delta}\|_2$. Now, both $\mathbf{a}_n^*$ and $\widetilde{\mathbf{a}}_n^*$ must satisfy the following inequalities

$$0 \leq (\Phi^*(\mathbf{q}) - \Phi^*(\mathbf{0}))(\mathcal{F}(\Phi^*(\mathbf{0}))) \tag{16}$$

$$0 \leq (\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q}))(\mathcal{F}(\Phi^*(\mathbf{q})) + \mathbf{q}) \tag{17}$$

where $\mathbf{0} = (\mathbf{0}_K)_1^N$ and $\mathbf{0}_K$ is the $K \times 1$ all zero vector. By some rearrangements of the above inequalities, we have

$$(\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q}))(\mathcal{F}(\Phi^*(\mathbf{0})) - \mathcal{F}(\Phi^*(\mathbf{q}))) \leq (\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q}))\mathbf{q} \tag{18}$$

Since $\Phi^*(\mathbf{q})$ is the co-coercive function of $\mathbf{q}$ (Proposition 2.3.11 in [18]), the left hand side of (18) is always less than $c_{\text{sm}}\|\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q})\|^2$. Using Schwartz inequality for the right hand side, we have

$$\|(\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q}))\mathbf{q}\|_2 \leq \|\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q})\|_2\|\mathbf{q}\|_2. \tag{19}$$

Since $\Phi^*(\mathbf{0})$ and $\Phi^*(\mathbf{q})$ correspond to $\mathbf{a}_n^*$ and $\widetilde{\mathbf{a}}_n^*$, respectively, (8) can be obtained.

Appendix C
Proof of Proposition 1

1) For the utility function (10), the NE is obtained using $AVI(\mathcal{A}, (\mathbf{w}_n + \mathbf{M}_n(\mathbf{a}_n))_{n=1}^N)$ where $\mathbf{w}_n = (\alpha_n^k (F_{nn}^k)^\theta)_{k=1}^K$, and $\mathbf{M}_n = (\frac{F_{mn}^k \alpha_m^n}{(F_{nn}^k)^{1+\frac{1}{\theta}}})_{k=1}^K$ [24].

2) Recall that any mapping $\widetilde{\mathbf{M}}(\mathbf{a}_n) = (\widetilde{\mathbf{M}}_n(\mathbf{a}_n))_{n=1}^N$ is said to be block contraction with module α, if there exists $\alpha \in [0, 1)$ such that $\forall \mathbf{a}_n^{(1)}, \mathbf{a}_n^{(2)} \in \mathcal{A}$ $\|\widetilde{\mathbf{M}}(\mathbf{a}_n^{(1)}) - \widetilde{\mathbf{M}}(\mathbf{a}_n^{(2)})\|_2 \leq \alpha\|\mathcal{A}_{-n}^{(1)} - \mathcal{A}_{-n}^{(2)}\|_2$. Using best response algorithm for the RACG, we have

$$\begin{aligned}
e_{\text{BR}_n} &\triangleq \| \mathbf{w}_n - \widetilde{\mathbf{M}}_n(\mathcal{A}_{-n}^{(1)}) - \mathbf{w}_n + \widetilde{\mathbf{M}}_n(\mathcal{A}_{-n}^{(2)}) \|_2 \leq \\
&\|[\sum_{m \neq n} \frac{F_{mn}^k \alpha_m^n}{(F_{nn}^k)^{1+\frac{1}{\theta}}}(a_{mn}^{k(1)} - a_{mn}^{k(2)}) + \boldsymbol{\epsilon}_n\| \|a_{-n}^1\|_2 - \|a_{-n}^2\|_2\|_2 \\
&\leq \sum_{m \neq n}(\max \frac{F_{mn}^k \alpha_m^n}{(F_{nn}^k)^{1+\frac{1}{\theta}}})\|\mathbf{a}_{-n}^{(1)} - \mathbf{a}_{-n}^{(2)}\|_2 + \\
&\quad \boldsymbol{\epsilon}_n\| \|a_{-n}^1\|_2 - \|a_{-n}^2\|_2\|_2 \leq \\
&\sum_{m \neq n}(\max \frac{F_{mn}^k \alpha_m^n}{(F_{nn}^k)^{1+\frac{1}{\theta}}}) + \|\boldsymbol{\epsilon}_n\|_\infty)\|\mathcal{A}_{-n}^{(1)} - \mathcal{A}_{-n}^{(2)}\|_2.
\end{aligned}$$

The above inequality can be written in vector form for all users as $\mathbf{0} \leq \mathbf{e}_{\text{BR}} \leq (\mathbf{M}^{\max} + \mathbf{E})\mathbf{e}$, where $\mathbf{e}_{\text{BR}} = [e_{\text{BR}_1}, \cdots, e_{\text{BR}_N}]$ and $\mathbf{e} = [\|\mathbf{a}_{-1}^{(1)} - \mathbf{a}_{-1}^{(2)}\|_2, \cdots, \|\mathbf{a}_{-N}^{(1)} - \mathbf{a}_{-N}^{(2)}\|_2]$. Using Schwartz inequality for any weighted norm, we write

$$\|\mathbf{e}_{\text{BR}}\|_2 \leq \|(\mathbf{M}^{\max} + \mathbf{E})\mathbf{e}\|_2 \leq \|\mathbf{M}^{\max} + \mathbf{E}\|_2\|\mathbf{e}\|_2. \tag{20}$$

Since $\|\mathbf{M}^{\max} + \mathbf{E}\|_2 < 1$, the best response algorithm for the RACG is contraction mapping, and therefore the RACG has a unique RNE.

3) The first part can be obtained the same as in Part 2 of Theorem 1. For the upper bound See Appendix E in [20].

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