Robust Stackelberg Communications Games

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Abstract

This paper studies multi-user communication systems with two groups of users, namely leaders who possess system information, and followers who have no such information, by using the formulation of Stackelberg games. In such games, leaders play and choose their actions based on their information about the system, and followers choose their actions myopically, according to their observations of the aggregate impact of other users. However, obtaining exact information is not practical in communication systems. To study the effect of uncertainty and preserve the players' utilities under such conditions, we introduce the notion of robust equilibrium for Stackelberg games, where the leaders' information and the followers' observations are uncertain parameters. In such a framework, leaders and followers choose their actions via the worst-case robust optimization. We show that that uncertainty in the followers' parameters always increases the leaders' utilities and decreases the followers' utilities. Conversely, uncertainty in the leaders' information reduces the leaders' utilities and increases the followers' utilities. We validate our theoretical results by way of numerical calculations for the power control game in interference channels.

Index Terms

Robust game theory, resource allocation, Stackelberg games, worst-case robust optimization.

I. Introduction

Stackelberg games provide a framework to analyze and design hierarchical interactions among rational, self-interested decision makers (players) [1], [2]. These hierarchical, non-cooperative games consist of two groups of players: leaders, who have complete information about the environment, including the system and other players' strategies; and followers, who have no such information. Each leader selects its action by solving a bi-level optimization problem that seeks to maximize its utility subject to the followers' actions as estimated by that leader. The followers then select their actions according to their observations from the aggregate impact of other users.

Recently, the authors in [3]–[5] have formalized the power control problem in interference channels as Stackelberg games. In these papers, the utility of each user is its throughput and its action is its transmit power. The leaders' information include the direct and the interference channel gains of all users, and the impacts of users on each other are interference levels at their receivers. In this system model, the leaders first determine their transmit power levels. Next, the followers' receivers measure (observe) their interference levels and send it back to their transmitters. Finally, each follower's transmitter determines its transmit power based on the interference level received from its receiver. Each leader's utilization of its information increases its throughput and may increase the followers' throughput in some cases (see, e.g., [3]).

However, extracting accurate system information either by the leaders or the followers is a key practical issue in implementing Stackelberg games in communication systems. For example, in wireless communication systems, fading, channel noise, delay in feedback channels, and users' mobility introduce uncertainty in measurements. Consequently, uncertainty in these parameters causes the leaders and the followers not to reach their expected utilities; in other words it leads to unreliable communications for both of them. Hence, it is essential to consider uncertainty in parameters' values and apply the robust approach to avoid such variations in utility values and provide for reliable transmissions to the extent possible.

Robust optimization theory is a branch of applied mathematics that can be used to mitigate the impact of uncertainties on optimization problems. In this context, each uncertain parameter is modeled by the sum of its nominal (estimated) value and an additive error (the uncertain part) [6]. Next, the optimization problem with nominal values (referred to as the nominal optimization problem) is mapped to another optimization problem (called the robust counterpart) in which the uncertain parameter is a new optimization parameter [7], [8].

Generally, two basic approaches are applied for this mapping [6], [7], [9]: the Bayesian approach, where the statistics of error is considered and the utility is statistically guaranteed; and the worst-case approach, where the error is assumed to be bounded within a specific region (the uncertainty region) and the utility is guaranteed for any realization of error within this region. Both of these approaches have been applied in communications, economics, and mathematics to tackle uncertain parameters in Stacklberg games [10]–[14]. In this paper, we choose the worst-case approach to preserve the players' utilities under any condition of error in the uncertainty region. We follow the terminology of robust optimization theory and call the Stackelberg game and its equilibrium with nominal values as the nominal Stackelberg game (NSG), and the nominal Stackelberg equilibrium (NSE), respectively. When uncertain values are considered and robust optimization is applied, we refer to the Stackelberg game and its equilibrium as the robust Stackelberg game (RSG) and the robust Stackelberg equilibrium (RSE), respectively.

The closest work to this paper is [14], which considers one-leader one-follower Stackelberg game and assumes that the leader does not know the exact values of some parameters in estimating the follower's action. By minimizing the second order sensitivity function of the leader's utility with respect to uncertain parameters, the worst-case utility for the leader given its imperfect information is obtained in [14]. However, to implement the RSG in communication systems, we encounter a number of challenges such as: 1) What is the definition of the RSE if the leaders and followers have different uncertain parameters? 2) What is the performance of the system in the RSG compared with that in the NSG? 3) How can we overcome the additional computations by each leader for solving this bi-level robust optimization problem? 4) How can we generalize the RSG for multi-leader multi-follower communication scenarios?

To answer the above questions, we first distinguish between the players' uncertain parameters. As stated before, in Stackelberg games, actions of leaders and of followers are determined by the leaders' information and the followers' observations about the aggregate impact of other players, respectively. Consequently, we assume that the uncertain parameters include the leaders' information and the followers' observations, and consider two cases for the RSE. In Case 1, the followers' observations are noisy while the leaders possess complete and accurate information; and in Case 2, the leaders' information is uncertain and the followers' observations are noisy.

To evaluate the performance of the RSG as compared to that of the NSG, we consider two criteria: i) the difference between the players' strategies at the RSE and at the NSE, and ii) the difference between the players' utilities at the RSE and at the NSE. The results show that for Case 1, uncertainty in the followers' observations increases the leaders' utilities and decreases the followers' utilities. The leaders' strategies are increasing functions and the followers' strategies are decreasing functions of the size of the uncertainty region. In contrast, for Case 2, uncertainty in the leaders' information decreases their utilities and increases the followers' utilities. The leaders' strategies are decreasing functions and the followers' strategies are increasing functions of the size of the uncertainty region. For both of these two cases, we derive the conditions (in terms of system parameters and interactions among users) under which the social utility at the RSE increases as compared to that at the NSE.

In this paper, we derive the relation between the players' strategies at the RSE and at the NSE in terms of system parameters and bounds of the uncertainty region. Based on this relationship, we also show that the complexity of solving robust bi-level optimization problems is reduced considerably. We begin our analysis of the RSE for the one-leader one-follower communication scenario, and generalize it to the multi-leader multi-follower scenario.

The rest of this paper is organized as follows. In Section II, the system model and game formulation are presented, followed by consideration of uncertainty in parameter values in Section III. In Section IV, two cases for the RSE are considered and analyzed for the one-leader one-follower scenario. In Section V, we apply our analytical findings to power control games and obtain numerical results for the RSE. In Section VI, we extend our framework for the RSG to the multi-leader multi-follower scenario, followed by conclusions in Section VII.

II. System Model and Game Formulation

A. Network Model

Consider a set of communication resources divided into K orthogonal dimensions, e.g., frequency bands, time slots, and routes, which are shared between a set of N users. Each user consists of a transmitter and a receiver. The set of possible positive actions of user n over all dimensions is

$$\mathcal{A}_n = \{\mathbf{a}_n = (a_n^1, \cdots, a_n^K) | a_n^k \in [a_{n,k}^{\mathtt{min}}, a_{n,k}^{\mathtt{max}}]\},\ \forall n \in \mathcal{N},\ \forall k \in \mathcal{K},$$

where $\mathcal{K} = [1, \cdots, K]$ and $\mathcal{N} = [1, \cdots, N]$. As an example of such a communications system, consider transmissions in interference channels over K sub-channels. The achieved utility of user n is $v_n(\mathbf{a})$, where $\mathbf{a} = [\mathbf{a}_n, \mathbf{a}_{-n}]$ and $\mathbf{a}_{-n} = (\mathbf{a}_0, \cdots, \mathbf{a}_{n-1}, \mathbf{a}_{n+1}, \cdots, \mathbf{a}_N)$ is a vector of other users' actions except user n.

We assume that:

A1) The utility of each user is an increasing, twice differentiable, and concave function of $\mathbf{a}_n$;

A2) the utility function of user n is

$$v_n(\mathbf{a}_n, \mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n)) = \sum_{k=1}^{K} v_n^k(a_n^k, f_n^k(\mathbf{a}_{-n}, \mathbf{x}_n)), \tag{1}$$

where $\mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n) = [f_n^1(\mathbf{a}_{-n}, \mathbf{x}_n), \cdots, f_n^K(\mathbf{a}_{-n}, \mathbf{x}_n)]$ is the $1 \times K$ vector of the linear aggregate impact of other users on user n, where

$$f_n^k(\mathbf{a}_{-n}, \mathbf{x}_n) = \sum_{m \in \mathcal{N},\ m \neq n} a_m^k x_{nm}^k + y_n^k, \tag{2}$$

and $\mathbf{x}_n = [\mathbf{x}_{n1}, \cdots, \mathbf{x}_{n(n-1)}, \mathbf{x}_{n(n+1)}, \cdots, \mathbf{x}_{nN}, \mathbf{y}_n]$ denotes system parameters for user n, $\mathbf{x}_{nm}$ is the $1 \times K$ vector and x_{nm}^k represents system parameters between user m and user n in dimension k, and $\mathbf{y}_n = [y_n^1, \cdots, y_n^K]$ where y_n^k denotes the impact of system on user n in dimension k. For example, for transmit power control in interference channels, $f_n^k(\mathbf{a}_{-n}, \mathbf{x}_n)$ is the interference of other users on user n in sub-channel k, i.e., $f_n^k(\mathbf{a}_{-n}, \mathbf{x}_n) = \sum_{m \neq n} h_{nm}^k a_m^k + \sigma_n^k$, where h_{nm}^k is the channel gain between user m and user n in sub-channel k, a_n^k is the transmit power of user n in sub-channel k, and σ_n^k is the channel noise in sub-channel k of user n. In this example, the interference channel gains between users and noise in each sub-channel are considered as system parameters, i.e., $\mathbf{x}_{nm} = \mathbf{h}_{nm}$ where $\mathbf{h}_{nm} = [h^1_{nm}, \cdots, h^K_{nm}]$ and $y^k_n = \sigma^k_n$;

A3) the utility of user n is a decreasing function of $f^k_n(\mathbf{a}_{-n}, \mathbf{x}_n)$;

A4) the utility function of each user is twice differentiable over $\mathbf{a}_n$, and $\mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n)$.

Note that A1 is a commonly assumed in the communication systems [15], [16]. In multiuser communication, A2 and A3 are well known when users share the same resources and have a negative impact on each other [17]. A4 indicates the differentiability of utility of each user with respect to $\mathbf{a}_n$ and $\mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n)$. Therefore, the above assumptions are justifiable in practical communications systems.

This setup includes different game-theoretic formulations of communication systems such as additively coupled sum constrained games [17], which can model many communication systems, including cellular transmissions within a given cell, and adhoc wireless network transmissions. In this paper, we consider power control games in interference channels as an illustrative example, where the throughput of each user in the system is its utility, i.e., $v_n(\mathbf{a}_n, \mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n)) = \sum_{k=1}^K \log(1 + \frac{h^k_{nn} a^k_n}{f^k_n(\mathbf{a}_{-n}, \mathbf{x}_n)})$.

Information obtained by user n is denoted by $\mathcal{I}_n$, which may be empty or contain private information of other users such as the values of their utilities, or their parameters' values. The information set of each user may be different from the others. The users with empty and non-empty information sets are referred to as followers and leaders, respectively.

The value of $\frac{\partial v^k_n(a^k_n, f^k_n(\mathbf{a}_{-n}, \mathbf{x}_n))}{\partial a^k_n}$ is the rate of change of the utility of user n for its action, which has a positive value as per assumption A1. A larger value of $\frac{\partial v^k_n(a^k_n, f^k_n(\mathbf{a}_{-n}, \mathbf{x}_n))}{\partial a^k_n}$ means a larger rate of increase in the n^{th} user's utility for its action. For example, in the power control game, we have

$$\frac{\partial v^k_n(a^k_n, f^k_n(\mathbf{a}_{-n}, \mathbf{x}_n))}{\partial a^k_n} = \frac{h^k_{nn}}{f^k_n + h^k_{nn} a^k_n}, \quad \forall n \in \mathcal{N}, \quad \forall k \in \mathcal{K}. \tag{3}$$

When the the direct channel gain for user n is high, i.e., $h^k_{nn} \gg 1$, or when its measured interference is low, i.e., $f^k_n \ll 1$, this value is large. This means that a small change in the user's action causes a significant change in the user's utility. We consider the column gradient vector v_n for user n, represented by $\mathbf{J}^n_{\mathbf{a}_n} = \nabla_{\mathbf{a}_n} v_n(\mathbf{a}_n, \mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n))$ and refer to it as the direct rate of user n. Let

$$\mathbf{C}_{nm} = \mathbf{X}_{nm} \mathbf{J}^n_{\mathbf{f}_n}, \tag{4}$$

where $\mathbf{J}^n_{\mathbf{f}_n} = \nabla_{\mathbf{f}_n} v_n(\mathbf{a}_n, \mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n))$ and $\mathbf{X}_{nm} \triangleq \text{diag}\{(x^k_{nm})^K_{k=1}\}$. For example, in power control games,

$$C^k_{nm} = -\frac{h^k_{nn} h^k_{nm} a^k_n}{f^k_n(f^k_n + h^k_{nn} a^k_n)}, \quad \forall n \in \mathcal{N}, \quad \forall k \in \mathcal{K}, \tag{5}$$

where C_{nm}^k is the k^{th} element of $\mathbf{C}_{nm}$. Note that a higher value of h_{nm}^k leads to a higher value of C_{nm}^k, i.e., a higher impact of user m on user n. This example shows that $\mathbf{C}_{nm}$ is in fact the rate of decrease in the utility of user n caused by a corresponding increase in the action of user m. Hence, we refer to $\mathbf{C}_{nm}$ as the negative impact of user m on user n. In what follows, we use $\mathbf{C}_{nm}$ and $\mathbf{J}_{\mathbf{a}_n}^n$ to study the effect of robustness on the social utility at the RSE as compared to that at the NSE.

B. Game Formulation

Now we model the interaction between informed and uninformed users in communication systems by a Stackelberg game. The set of leaders and the set of followers in the Stackelberg game are denoted by $\mathcal{N}_{\text{L}} = \{1, \cdots, N_{\text{L}}\}$ and $\mathcal{N}_{\text{F}} = \{1, \cdots, N_{\text{F}}\}$, respectively, and $\mathcal{N} = \mathcal{N}_{\text{L}} \cup \mathcal{N}_{\text{F}}$ is the set of all players in the game, where

$$\boldsymbol{\mathcal{I}}_n = \quad \{(\mathcal{A}_m, v_m, \mathbf{X}_{mn}, \mathbf{X}_{mm}, \mathbf{X}_{nm})_{m \neq n, \forall m \in \mathcal{N}}\} \qquad \text{if} \quad n \in \mathcal{N}_{\text{L}},$$
$$\boldsymbol{\mathcal{I}}_n = \quad \emptyset \qquad \text{if} \quad n \in \mathcal{N}_{\text{F}},$$

where $\mathbf{X}_{mn} \triangleq \text{diag}\{(x_{mn}^k)_{k=1}^K\}$ and $\mathbf{X}_{mm} \triangleq \text{diag}\{(x_{mm}^k)_{k=1}^K\}$. In the Stackelberg game, leaders play their strategy first, followed by measuring the value of $\mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n)$ by the receiver of follower n and sending it to its corresponding transmitter to decide its action. The value of $\mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n)$ is the observation of follower n on the aggregate impact of other users, and the follower's optimization problem is

$$\max_{\mathbf{a}_n \in \mathcal{A}_n} v_n(\mathbf{a}_n, \mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n)), \quad \forall n \in \mathcal{N}_{\text{F}}. \tag{6}$$

The solution to (6) for user n represents its best response, denoted by $\mathbf{a}_n^*(\mathbf{a}_{-n})$. Since the followers are non-cooperative players, the Nash equilibrium (NE) of the game is $\mathbf{a}^{*\text{NE}} = (\mathbf{a}_1^*, \cdots, \mathbf{a}_{N_f}^*)$, which satisfies

$$v_n(\mathbf{a}_n^*, \mathbf{f}_n(\mathbf{a}_{-n}^*, \mathbf{x}_n)) \geq v_n(\mathbf{a}_n, \mathbf{f}_n(\mathbf{a}_{-n}^*, \mathbf{x}_n)), \qquad \mathbf{a}_n^* \in \mathcal{A}_n, \tag{7}$$

where $\mathbf{a}_{-n}^* = [\mathbf{a}_0^*, \cdots, \mathbf{a}_{n-1}^*, \mathbf{a}_{n+1}^*, \cdots, \mathbf{a}_N^*]$ for all $n \in \mathcal{N}$. When $\boldsymbol{\mathcal{I}}_n = \emptyset$ for all players, the game reduces to a strategic non-cooperative game.

To define the equilibrium of the Stackelberg game, we consider a one-leader Stackelberg game, where player 0 is the leader. The equilibrium in the Stackelberg game prescribes the optimal strategy for the leader when the followers play at their NE. For example, in a one-leader one-follower Stackelberg game, where player 1 is the follower, the action $\mathbf{a}_0^*$ is the leader's strategy at the Stackelberg equilibrium when for $\mathbf{a}_0 \in \mathcal{A}_0$ we have

$$v_0(\mathbf{a}_0^{*\text{NSE}}, \mathbf{f}_0(\mathbf{a}_1^*(\mathbf{a}_0^{*\text{NSE}}), \mathbf{x}_n)) \geq v_0(\mathbf{a}_0, \mathbf{f}_0(\mathbf{a}_1^*(\mathbf{a}_0), \mathbf{x}_n)). \tag{8}$$

In this case, the leader's optimization problem changes to the following bi-level optimization problem

$$\max_{\mathbf{a}_0 \in \mathcal{A}_0} v_0(\mathbf{a}_0, \mathbf{f}_0(\mathbf{a}_1, \mathbf{x}_n)), \quad \forall n \in \mathcal{N} \qquad (9)$$
$$\text{subject to:} \quad \max_{\mathbf{a}_1 \in \mathcal{A}_1} v_1(\mathbf{a}_1, \mathbf{f}_1(\mathbf{a}_0, \mathbf{x}_n)).$$

For the multi-follower scenario, let $\mathbf{a}^*_{-0}(\mathbf{a}_0) = [\mathbf{a}^*_1, \cdots, \mathbf{a}^*_{N-1}]$ be the NE strategy of the followers when the leader plays $\mathbf{a}_0$. The strategy profile $(\mathbf{a}_0^{*\text{NSE}}, \mathbf{a}_{-0}^{*\text{NSE}}(\mathbf{a}_0^{*\text{NSE}}))$ is the equilibrium of the Stackelberg game iff

$$v_0(\mathbf{a}_0^{*\text{NSE}}, \mathbf{f}_0(\mathbf{a}_{-0}^{*\text{NSE}}(\mathbf{a}_0^{*\text{NSE}}), \mathbf{x}_0)) \geq v_0(\mathbf{a}_0, \mathbf{f}(\mathbf{a}^*_{-0}(\mathbf{a}_0), \mathbf{x}_0)), \quad \forall \mathbf{a}_0 \in \mathcal{A}_0.$$

We denote the achieved utility of player n at the NSE and the social utility of the game by $\omega_n^{*\text{NSE}}$ and $\omega^{*\text{NSE}} = \sum_{n \in \mathcal{N}} \omega_n^{*\text{NSE}}$, respectively.

When the followers' game has multiple NEs, the NSE is more complicated as described in [2], [18]–[20]. In this paper, we restrict our study to the Stackelberg game with a unique NE in the followers' game. The uniqueness condition for this game is provided in Section VI. In what follows, we consider uncertainty in system parameters, and introduce different types of RSE and the robust counterpart optimization problems for both leaders and followers.

III. Uncertain Parameters

As stated before, both the followers' observations and the leaders' information sets are uncertain parameters in the considered communication scenario. In the following subsections, we define the followers' uncertain observations and the leaders' uncertain information set which are called noisy observations and incomplete information sets, respectively.

A. *Noisy Observations*

Consider the uncertain value of $\mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n)$ as a noisy observation by user n of the impact of the other users, modeled by the sum of its nominal value and an error [21], i.e.,

$$\widetilde{\mathbf{f}}_n(\mathbf{a}_{-n}, \mathbf{x}_n) = \mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n) + \widehat{\mathbf{f}}_n(\mathbf{a}_{-n}, \mathbf{x}_n), \qquad (10)$$

where $\widetilde{\mathbf{f}}_n(\mathbf{a}_{-n}, \mathbf{x}_n) = [\widetilde{f}_n^1(\mathbf{a}_{-n}, \mathbf{x}_n), \cdots \widetilde{f}_n^K(\mathbf{a}_{-n}, \mathbf{x}_n)]$, $\mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n) = [f_n^1(\mathbf{a}_{-n}, \mathbf{x}_n), \cdots f_n^K(\mathbf{a}_{-n}, \mathbf{x}_n)]$, and $\widehat{\mathbf{f}}_n(\mathbf{a}_{-n}, \mathbf{x}_n) = [\widehat{f}_n^1(\mathbf{a}_{-n}, \mathbf{x}_n), \cdots \widehat{f}_n^K(\mathbf{a}_{-n}, \mathbf{x}_n)]$ are the exact value, the nominal value, and the error in the observation of user n, respectively. In the worst-case robust optimization theory, the error in noisy observations is assumed to be bounded in a closed uncertainty region [21], [22] defined by

$$\Re_n(\mathbf{a}_{-n}) = \{\widetilde{\mathbf{f}}_n(\mathbf{a}_{-n}, \mathbf{x}_n) \in \Re_n(\mathbf{a}_{-n}) | \|\widehat{\mathbf{f}}_n(\mathbf{a}_{-n}, \mathbf{x}_n)\|_2 \leq \varepsilon_n\}, \quad \forall n \in \mathcal{N}_\text{F}, \qquad (11)$$

where ε_n is the bound on the uncertainty region, and $\|.\|_2$ is the ellipsoid norm. Uncertainty in communications system has a probabilistic nature, and the uncertainty region is formed in such a way that with a given potability, all realizations of error fall in that region. In the literature, the ellipsoid norm has been commonly applied to model the uncertainty region in wireless channels [6], [23]–[27]. Following on them, we also choose the ellipsoid norm in this paper.

The noisy observation is considered as a new optimization parameter in the utility of each users [21]. The new utility function of user n when uncertainty is considered is $u_n(\mathbf{a}_n, \widetilde{\mathbf{f}}_n)$, which satisfies

$$u_n(\mathbf{a}_n, \widetilde{\mathbf{f}}_n(\mathbf{a}_{-n}, \mathbf{x}_n))|_{\varepsilon_n=0} = v_n(\mathbf{a}_n, \mathbf{f}_n(\mathbf{a}_{-n}, \mathbf{x}_n)). \tag{12}$$

Consequently, the followers' optimization problem changes to

$$\max_{\mathbf{a}_n \in \mathcal{A}_n} \min_{\widetilde{\mathbf{f}}_n(\mathbf{a}_{-n}, \mathbf{x}_n) \in \Re_n(\mathbf{a}_{-n})} u_n(\mathbf{a}_n, \widetilde{\mathbf{f}}_n(\mathbf{a}_{-n}, \mathbf{x}_n)), \quad \forall n \in \mathcal{N}_{\mathrm{F}}. \tag{13}$$

The robust Nash equilibrium (RNE) of this game by assuming $\boldsymbol{\mathcal{I}}_n = \emptyset$ for all users is $\widetilde{\mathbf{a}}^* = (\widetilde{\mathbf{a}}_0^*, \cdots, \widetilde{\mathbf{a}}_{N-1}^*)$ [21], [22] iff

$$\min_{\widetilde{\mathbf{f}}_n(\mathbf{a}^*_{-n}, \mathbf{x}_n) \in \Re_n(\mathbf{a}_{-n})} u_n(\mathbf{a}^*_n, \widetilde{\mathbf{f}}_n(\mathbf{a}^*_{-n}, \mathbf{x}_n)) \geq \min_{\widetilde{\mathbf{f}}_n(\mathbf{a}^*_{-n}, \mathbf{x}_n) \in \Re_n(\mathbf{a}_{-n})} u_n(\mathbf{a}_n, \widetilde{\mathbf{f}}_n(\mathbf{a}^*_{-n}, \mathbf{x}_n)), \quad \forall \mathbf{a}_n \in \mathcal{A}_n. \tag{14}$$

B. *Incomplete Information Set*

We refer to the leaders' information set with uncertain parameters as an incomplete information set where $\mathbf{X}_{n_{\mathrm{F}} n_{\mathrm{L}}}$ is the uncertain parameter. Obtaining the value of this parameter is more challenging as compared to obtaining the values of other parameters in communication systems. For example, in the power control game, the follower's transmitter sends a pilot signal to its receiver for estimating its direct channel gain. The leader can extract $\mathbf{H}_{n_{\mathrm{F}} n_{\mathrm{F}}}$ by listening to the follower's feedback channel. The leader can also estimate the value of $\mathbf{H}_{n_{\mathrm{L}} n_{\mathrm{F}}}$ by listening to this pilot signal. However, since there is no pilot signal from the follower's receiver, the leader cannot estimate the value of $\mathbf{H}_{n_{\mathrm{F}} n_{\mathrm{L}}}$. In the worst-case approach, the uncertain information is modeled by the sum of its nominal value and the uncertain part bounded in the uncertainty region defined by

$$\mathcal{R}_{\mathbf{X}_{n_{\mathrm{F}} n_{\mathrm{L}}}} = \{\widetilde{\mathbf{X}}_{n_{\mathrm{F}} n_{\mathrm{L}}} \,|\, \|\widehat{\mathbf{X}}_{n_{\mathrm{F}} n_{\mathrm{L}}}\|_2 = \|\widetilde{\mathbf{X}}_{n_{\mathrm{F}} n_{\mathrm{L}}} - \mathbf{X}_{n_{\mathrm{F}} n_{\mathrm{L}}}\|_2 \leq \delta_{n_{\mathrm{F}} n_{\mathrm{L}}}\}, \quad n_{\mathrm{F}} \in \mathcal{N}_{\mathrm{F}}, \quad n_{\mathrm{L}} \in \mathcal{N}_{\mathrm{L}}. \tag{15}$$

In the following, we refer to the leader's information set without uncertainty as the complete information set.

IV. Robust Stackelberg Equilibrium

We now discuss different forms of RSG in the following two cases.

Case 1: The leaders possess complete information, and the followers possess noisy observations;

Case 2: The leaders possess incomplete information and the followers possess noisy observations.

We begin by considering the one-leader one-follower scenario, and extend our analysis of the RSE for the multi-leader multi-follower scenario in Section VI.

A. Analysis of RSE for Case 1

In this case, the leader's information set denoted by $\boldsymbol{\mathcal{I}}_0^{\text{RSE1}}$ is

$$\boldsymbol{\mathcal{I}}_0^{\text{RSE1}} = \{(\mathcal{A}_m, v_m, \mathbf{X}_{mn}, \mathbf{X}_{mm}, \mathbf{X}_{nm}, \Re_m(\mathbf{a}_{-m}))_{m\neq n}\}, \quad \forall m \in \mathcal{N}. \tag{16}$$

The leader also knows that the follower's optimization problem is the same as (13). Hence, the leader's bi-level optimization problem is

$$\begin{aligned} &\max\nolimits_{\mathbf{a}_0\in\mathcal{A}_0} \quad v_0(\mathbf{a}_0, \mathbf{f}_0(\mathbf{a}_1, \mathbf{x}_{01})) \\ \text{subject to:} \quad &\max\nolimits_{\mathbf{a}_n\in\mathcal{A}_n} \min_{\widetilde{\mathbf{f}}_1\in\Re_1(\mathbf{a}_{-1})} u_1(\mathbf{a}_1, \widetilde{\mathbf{f}}_1(\mathbf{a}_0, \mathbf{x}_{10})). \end{aligned} \tag{17}$$

The best response of (13) to the leader's action is denoted by $\widetilde{\mathbf{a}}_1^*(\mathbf{a}_0)$, and the RSE for Case 1 denoted by $\mathbf{a}_0^{*\text{RSE1}}$ satisfies

$$v_0(\mathbf{a}_0^{*\text{RSE1}}, \widetilde{\mathbf{f}}_0(\widetilde{\mathbf{a}}_1^*(\mathbf{a}_0^{*\text{RSE1}}), \mathbf{x}_0)) \geq v_0(\mathbf{a}_0, \widetilde{\mathbf{f}}_0(\widetilde{\mathbf{a}}_1^*(\mathbf{a}_0), \mathbf{x}_0)). \tag{18}$$

Remark 1. The RSE of Case 1 exists since: 1) (13) is concave with respect to $\mathbf{a}_1(\mathbf{a}_0)$ for any fixed action of the leader, and is a decreasing function of $\mathbf{f}_1(\mathbf{a}_0, \mathbf{x}_1)$, 2) The two sets $\mathcal{A}_1$ and $\Re_1(\mathbf{a}_{-1})$ are convex, bounded, and disjoint. Consequently, there always exists a saddle point of (13) [23], which is the solution to (13).

Now we characterize and compute the RSE, which is the most difficult part due to the computational complexity of (17). In what follows, for notational convenience, we omit the arguments of $\mathbf{f}_1(\mathbf{a}_0, \mathbf{x}_1)$ and $\mathbf{f}_0(\mathbf{a}_0, \mathbf{x}_0)$.

Lemma 1. The uncertain observation for the follower's optimization problem is

$$\widetilde{\mathbf{f}}_1^* = \mathbf{f}_1 - \varepsilon_1 \boldsymbol{\vartheta}_1, \tag{19}$$

where $\widetilde{\mathbf{f}}_1^* = [\widetilde{f}_1^{1*}, \cdots, \widetilde{f}_1^{K*}]$, $\boldsymbol{\vartheta}_1 = [\vartheta_1^1, \cdots, \vartheta_1^K]$, and ϑ_1^k is defined by

$$\vartheta_1^k = \frac{\frac{\partial u_1^k(\mathbf{a}_1, \widetilde{\mathbf{f}}_1)}{\partial f_1^k}}{\sqrt{\sum_{k=1}^K (\frac{\partial u_1^k(\mathbf{a}_1, \widetilde{\mathbf{f}}_1)}{\partial f_1^k})^2}}. \tag{20}$$

Proof: See Appendix A. ■

Using (19) in problem (17) removes the uncertainty region from the leader's bi-level optimization problem, which is simplified to

$$\max_{\mathbf{a}_0 \in \mathcal{A}_0} v_0(\mathbf{a}_0, \mathbf{f}_0), \quad \forall n \in \mathcal{N} \tag{21}$$
$$\text{subject to:} \quad \max_{\mathbf{a}_1 \in \mathcal{A}_1} v_1(\mathbf{a}_1, \widetilde{\mathbf{f}}_1^*).$$

By this reformulation, we can derive the changes in the the leader's strategy and in the follower's strategy at the RSE as compared to those at the NSE for Case 1.

Proposition 1. For Case 1 in the RSG, the leader's action is an increasing function and the follower's action is a decreasing function of ε_1, and are obtained by

$$\mathbf{a}_1^{*\text{RSE1}} = \mathbf{a}_1^{*\text{NSE}} - \varepsilon_1 \times ((\mathbf{J}^1_{\mathbf{a}_1\mathbf{a}_1})^{-1}\mathbf{J}^1_{\mathbf{a}_1\mathbf{f}_1} \times \vartheta_1^{\text{T}})^{\text{T}}, \tag{22}$$

$$\mathbf{a}_0^{*\text{RSE1}} = \mathbf{a}_0^{*\text{NSE}} + \varepsilon_1 \times ((\mathbf{J}^0_{\mathbf{a}_0\mathbf{a}_0})^{-1}\mathbf{J}^0_{\mathbf{a}_0,\mathbf{f}_0}\boldsymbol{X}_{01}(\mathbf{J}^1_{\mathbf{a}_1\mathbf{a}_1})^{-1}\mathbf{J}^1_{\mathbf{a}_1\mathbf{f}_1} \times \vartheta_1^{\text{T}})^{\text{T}}, \tag{23}$$

where $\mathbf{J}^n_{\mathbf{f}_n\mathbf{a}_n} = \nabla_{\mathbf{f}_n\mathbf{a}_n} v_n(\mathbf{a}_n\mathbf{f}_n)$ and $\mathbf{J}^n_{\mathbf{a}_n\mathbf{a}_n} = \nabla^2_{\mathbf{a}_n} v_n(\mathbf{a}_n, \mathbf{f}_n)$.

Proof: See Appendix B. ■

From Proposition 1, the solution to the robust problem (17) can be obtained via the NSE and the bound of the uncertainty region in (22) and (23). In this way, the computational complexity to solve (17) is reduced. From (21) and Proposition 1, the changes in utilities of the leader and the follower are obtained below.

Proposition 2. For Case 1 in the RSG, 1) for any realization of the follower's noisy observation, we have

$$\omega_0^{*\text{NSE}} \le \omega_0^{*\text{RSE1}}, \quad \omega_1^{*\text{RSE1}} \le \omega_1^{*\text{NSE}},$$

where $\omega_n^{*\text{RSE1}}$ is the achieved utility of player n at the RSE for Case 1. 2) The social utility of the game increases as compared to that of the NSG, i.e., $\omega^{*\text{RSE1}} > \omega^{*\text{NSE}}$ where $\omega^{*\text{RSE1}}$ is the social utility of game in RSE1, if

$$\text{C1}: \ |\mathbf{C}_{10}| < |\mathbf{J}^0_{\mathbf{a}_0}|, \qquad \text{C2}: \ |\mathbf{J}^1_{\mathbf{a}_1}| < |\mathbf{C}_{01}|,$$

where $|\mathbf{q}|$ is the absolute value of the elements of $\mathbf{q}$.

Proof: See Appendix C. ■

From Proposition 2, uncertainty in the follower's observation increases the leader's utility. In contrast, the follower achieves a smaller utility at the RSE for Case 1 as compared to that at the NSE. Interestingly, the social utility at the RSE increases as compared to that at the NSE when C1 and C2 hold. The constraints C1 and C2 can be interpreted as follows. The negative impact of the leader on the follower is less than the leader's direct rate and the follower's direct rate is less than its negative impact on the leader. In other words, the increase in the leader's utility is higher than the decrease in the follower's utility when the leader's strategy increases and the follower's strategy decreases; resulting in an increase in the social utility.

B. Analysis of RSE for Case 2

For case 2, the leader's incomplete information set, denoted by $\widetilde{\mathcal{I}}_0^{\text{RSE2}}$, is

$$\widetilde{\mathcal{I}}_0^{\text{RSE2}} = \{(\mathcal{A}_m, v_m, \widetilde{\mathbf{X}}_{mn}, \mathbf{X}_{mm}, \mathbf{X}_{nm}, \Re_m(\mathbf{a}_{-m}))_{m \neq n}\}, \quad \forall m \in \mathcal{N}, \tag{24}$$

where $\widetilde{\mathbf{X}}_{10}$ is the uncertain parameter in the uncertainty region (15). Using the worst-case optimization, the leader's bi-level optimization problem is changed to

$$\begin{aligned} &\max_{\mathbf{a}_0 \in \mathcal{A}_0} \min_{\widetilde{\mathbf{X}}_{10} \in \mathcal{R}_{\mathbf{X}_{10}}} \; v_0(\mathbf{a}_0, \mathbf{f}_0) \\ \text{subject to:} \quad &\max_{\mathbf{a}_1 \in \mathcal{A}_1} \min_{\mathbf{f}_1 \in \mathcal{R}_1} u_1(\mathbf{a}_1, \mathbf{f}_1). \end{aligned} \tag{25}$$

According to (25), the leader cannot accurately evaluate its impact on the follower. Since $\mathbf{f}_1$ is a linear function of $\mathbf{X}_{10}$, and the leader considers the worst-case instance in the uncertainty region to obtain the solution of the follower, we have:

A5) For Case 2, the negative impact of the leader on the follower is a decreasing function of δ_{10}, i.e., $\nabla_{\delta_{10}} \mathbf{f}_1 < 0$.

Remark 2. The RSE in Case 2 always exists, because: 1) $\mathcal{R}_{\mathbf{X}_{10}}$, $\mathcal{R}_1$, $\mathcal{A}_0$, and $\mathcal{A}_1$ are compact and closed sets, 2) for any realization of $\widetilde{\mathbf{X}}_{10} \in \mathcal{R}_{\mathbf{X}_{10}}$, $\mathcal{R}_1$ is closed and convex. Consequently, for any value of the leader's uncertain information and strategy, the follower has a feasible strategy.

While the condition for the existence of the RSE for Case 2 can be derived easily, solving (25) is significantly more complex as compared to (17). In what follows, we discuss the relationship between the RSE and the NSE for Cases 1 and 2.

Proposition 3. The leader's utility at the RSE in Case 2 is always less than the leader's utility in Case 1, i.e., $\omega_0^{*\text{RSE2}} \leq \omega_0^{*\text{RSE1}}$, where $\omega_n^{*\text{RSE2}}$ is the achieved utility of user n at the RSE in Case 2.

Proof: See Appendix D. ■

Proposition 3 shows that the leader's incomplete information set always decreases the leader's utility as compared to Case 1. Next, we compare the leader's and the follower's utilities at the RSE for Case 2 with those at the NSE.

Proposition 4. For Case 2 in the RSG: 1) The leader's strategy is a decreasing function of δ_{10} and the follower's strategy is an increasing function of δ_{10}, 2) for all realizations of the leader's incomplete information set, we have

$$\omega_0^{*\text{RSE2}} \leq \omega_0^{*\text{NSE}}, \qquad \omega_1^{*\text{NSE}} \leq \omega_1^{*\text{RSE2}}.$$

3) the social utility of the RSG is higher than that of the NSG, i.e., $\omega^{*\text{RSE2}} \geq \omega^{*\text{NSE}}$, if

$$\text{C3}: \quad |\mathbf{J}^0_{\mathbf{a}_0}| < |\mathbf{C}_{10}|, \quad \text{C4}: \quad |\mathbf{J}^1_{\mathbf{a}_1}| > |\mathbf{C}_{01}|.$$

Proof: Appendix E. ∎

From Proposition 4, uncertainty in the leader's information set always decreases the leader's utility as compared to that at the NSE. In contrast, the follower achieves a higher utility as compared to that at the NSE. In this case, when the follower's direct rate is greater than its negative impact on the leader (i.e., C3), and the leader's direct rate is less than its negative impact on the follower (i.e., C4), the social utility at the RSE for Case 2 increases as compared to that at the NSE.

An interesting interpretation arises when comparing C1 with C3 and C2 with C4. These comparisons indicate that C1-C2 are the dual of C3-C4. In Case 1, a higher social utility can be achieved if the increase in the leader's utility is higher than the decrease in the follower's utility. In contrast, in Case 2, a higher utility can be achieved when the increase in the follower's utility is higher than the decrease in the leader's utility. The variations of utilities of the leader and the follower at the RSE as compared to those at the NSE for Cases 1 and 2 are summarized in Table I.

Note that the implementation of above cases in practice does not need the leader and the follower to be synchronized. For example, if the follower plays first, the leader chooses its action based on its information set without considering the follower's action. Then the follower observes the leader's impact and plays based on its observation. Therefore, the actions of the leader and the follower always converge to the NSE and the RSE regardless of synchronization between them.

V. Illustrative Example

We now validate the above for a power control game where C1-C4 are simplified taking into account the channel gains between users. In the power control game, the information set for Case 1 is $\mathcal{I}_0^{\text{RSE1}} = \{(\mathcal{A}_m, v_m, \widetilde{\mathbf{h}}_{mn}, \mathbf{h}_{mm}, \mathbf{h}_{nm}, \Re_m(\mathbf{a}_{-m}))_{m\neq n}, \forall m \in \mathcal{N}\}$. In this case, the solution to the follower's optimization problem when uncertainty is considered is

$$\widetilde{f}_1^{*k} = f_1^k + \varepsilon_1 \times \frac{\frac{a_1^k h_{11}^k}{\sigma_1^k + a_0^k h_{01}^k}}{\sqrt{(\sum_{k=1}^{K} \frac{a_1^k h_{11}^k}{\sigma_1^k + a_0^k h_{01}^k})^2}}, \quad \forall k \in \mathcal{K}. \tag{26}$$

To illustrate the results of Propositions 2 and 4, we simulate a single carrier power control game in which a Rayleigh fading channel modeled in [3] is assumed, $K = 1$, $a_{nk}^{\max} = 10$ dB, $a_{nk}^{\min} = 0$ dB and $\sigma_n = 0.01$. Fig. 1 shows variations in utilities of the follower and the leader on the Pareto boundary of the power control game. By increasing ε_1, the leader's utility increases as expected from Proposition 2. In contrast, by increasing the value of δ_{10}, the follower's utility increases and the value of the leader's utility decreases as expected from Proposition 4.

To provide a practical insight into C1 and C2 for the power control game, we express these conditions only in terms of channel gains. The exact expressions of C1 and C2 for the power control game are

$$\text{C1 for power control games}: \quad \frac{h_{10}^k h_{11}^k a_1^k}{f_1^k \times (f_1^k + h_{11}^k a_1^k)} < \frac{h_{00}^k}{f_0^k + h_{00}^k a_0^k}, \qquad \forall k \in \mathcal{K}, \tag{27}$$

$$\text{C2 for power control games}: \quad \frac{h_{01}^k h_{00}^k a_0^k}{f_0^k \times (f_0^k + h_{00}^k a_0^k)} > \frac{h_{11}^k}{f_1^k + h_{11}^k a_1^k}, \qquad \forall k \in \mathcal{K}. \tag{28}$$

To simplify the above conditions, we consider the following three scenarios based on signal-to-interference-plus-noise ratios (SINRs) of the leader and the follower.

Scenario 1. High SINR, i.e., $h_{00}^k a_0^k \gg h_{01}^k a_1^k + \sigma_0^k$ and $h_{11}^k a_1^k \gg h_{10}^k a_0^k + \sigma_1^k$, where C1 and C2 simplify to

$$h_{10}^k < h_{01}^k. \tag{29}$$

Scenario 2. Low SINR, i.e., $h_{00}^k a_0^k \ll h_{01}^k a_1^k + \sigma_0^k$ and $h_{11}^k a_1^k \ll h_{10}^k a_0^k + \sigma_1^k$, where the social utility increases when

$$h_{00}^k > h_{01}^k \quad \text{and} \quad h_{11}^k < h_{10}^k. \tag{30}$$

Scenario 3. Moderate SINR, when induced interferences of the leader and the follower on each other are close, i.e., $f_1^k \approx f_0^k$, and C1 and C2 change to

$$\frac{h_{00}^k}{h_{01}^k} > \frac{h_{11}^k a_1^k}{h_{10}^k a_0^k + \sigma_1^k} \quad \text{and} \quad \frac{h_{11}^k}{h_{10}^k} < \frac{h_{00}^k a_0^k}{h_{01}^k a_1^k + \sigma_0^k} \quad \forall k \in \mathcal{K}.$$

When channel noise is much less than interference from other users, i.e., $h_{10}^k a_0^k \gg \sigma_1^k$, and transmit power levels of the leader and the follower are close, the above conditions are simplified to

$$h_{00}^k h_{10}^k > h_{11}^k h_{10}^k, \qquad \forall k \in K. \tag{31}$$

Using (29)-(31), we can predict how the social utility increases or decreases under the given channel conditions for Case 1. To gain an insight into the behavior of Stackelberg games in interference channels, we investigate the effect of these conditions on the leader's utility and on the follower's utility. Consider $d_n^{\text{RSE1}} = \frac{\omega_n^{*\text{RSE1}} - \omega_n^{*\text{NSE}}}{\omega_n^{*\text{NSE}}}$ as the change in the utility of player n at the RSE for Case 1 as compared to the same at the NSE, and $d^{\text{RSE1}} = \frac{\omega^{*\text{RSE1}} - \omega^{*\text{NSE}}}{\omega^{*\text{NSE}}}$ as the change in the social utility at the RSE for Case 1 as compared to that at the NSE. A larger value of d_n^{RSE1} indicates a larger increase in the utility of player n for Case 1. For Case 2, we have $d_n^{\text{RSE2}} = \frac{\omega_n^{*\text{RSE2}} - \omega_n^{*\text{NSE}}}{\omega_n^{*\text{NSE}}}$ and $d^{\text{RSE2}} = \frac{\omega^{*\text{RSE2}} - \omega^{*\text{NSE}}}{\omega^{*\text{NSE}}}$ for user n and for social utility, respectively.

In Table II, we show the values of d_0^{RSE1}, d_1^{RSE1} and d^{RSE1} for Scenarios 1-3, respectively. Simulation parameters are the same as in the pervious simulations except that the channel is four-ray Rayleigh [3] and $K = 16$. Note the increase in the leader's utility and the decrease in the follower's utility in terms of ε_1, as expected from Proposition 2. When (29)-(31) hold, the social utility is increased. In contrast, when (29)-(31) do not hold, the social utility is less than that at the NSE. However, in Scenario 1, the increase in the social utility is not considerable when (29) holds. In Scenarios 2 and 3, when (30) and (31) hold, the leader's utility and the social utility are increased considerably. For example, when (30) does not hold, the value of d_0^{RSE1} is reduced to 40% from 220%, and d^{RSE1} is reduced to around -1% from 100%. Hence, accuracy in the information set is immensely beneficial to the leader and to the system when (30) holds. The same is true for Scenario 3.

For Case 2, the uncertain parameter is $\mathbf{h}_{10}$. For the above scenarios, C_3 and C_4 change to

$$\text{Scenario 1:} \quad \Longrightarrow \quad h_{10}^k > h_{01}^k \tag{32}$$

$$\text{Scenario 2:} \quad \Longrightarrow \quad h_{00}^k < h_{01}^k, \quad h_{11}^k > h_{10}^k \tag{33}$$

$$\text{Scenario 3:} \quad \Longrightarrow \quad h_{00}^k h_{10}^k < h_{11}^k h_{10}^k, \quad \forall k. \tag{34}$$

The effects of increasing the value of δ_{01} on d_0^{RSE2}, d_1^{RSE2}, and d^{RSE2} are shown in Table III. As expected from Proposition 4, the leader's utilities in all cases are less than those at the NSE, while the follower's utility is increased by increasing the value of δ_{01}. In Table III, when the conditions (32)-(34) do not hold, the social utility decreases. In contrast, when (32)-(34) hold, the social utility increases by increasing δ_{10}. Also, d_1^{RSE2} decreases from 1.5% to 1% when (32) holds as compared to the case when it does not hold. The same is observable by comparing d_1^{RSE2} in different situations. We conclude that in all scenarios, the increase in the follower's utility is reduced when (32)-(34) hold compared to the case when (32)-(34) do not hold, and the increase in the social utility is insignificant.

A. Practical Remarks

Comparing Tables II and III shows that in Case 1, the leader's utility and the social utility at the RSE are significantly higher than those at the NSE. This is in contrast to Case 2, where the increase in the social utility is moderate, and the increase in the follower's utility is insignificant. For example, in Scenario 2, the leader's utility and the social utility at the RSE are up to 200% higher than those at the NSE for Case 1, whereas in Scenario 2, the follower's utility and the social utility at the RSE are up to 10% higher than those at the NSE for Case 2. Therefore, the leader's complete information set is more effective in increasing the social utility. This analysis indicates that a scheme for obtaining accurate information for the leader is very desirable. The above are useful in designing systems with two or more leaders. Consider the case that leaders want to increase their social utility, but some of them encounter uncertainty in their parameters. If (29)- (31) hold for one leader who has complete information, it plays as a leader and the others play as followers. In this way, the social utility of leaders increases considerably.

B. Power Control with Bounded Transmit Power

When the sum of transmit power levels of each player over all sub-channels is upper bounded to $P_n^{\max}$, i.e.,

$$\sum_{k=1}^{K} a_n^k \leq P_n^{\max}, \tag{35}$$

the players' strategies are nonlinear functions of their observations [3], [29]. Therefore, we cannot directly use Propositions 1-4. The performance of the game at the RSE for (35) in terms of the allocated transmit power over different sub-channels is shown in Figs. 2-4. The corresponding leader's utility and follower's utility are summarized in Table IV. Simulation parameters are the same as in Table II except for the power mask, which is $P_n^{\max} = a_{nk}^{\max} = 200$ dB.

Let $\mathcal{K}_n^{\text{NSE}} \subseteq \mathcal{K}$ and $\mathcal{K}_n^{\text{RSE1}} \subseteq \mathcal{K}$ be the sets of sub-channels utilized by user n at the NSE and at the RSE for Case 1, respectively. Also, let $\mathcal{L}_{nm}^{\text{NSE}} = \mathcal{K}_n^{\text{NSE}} \cap \mathcal{K}_m^{\text{NSE}}$ and $\mathcal{L}_{nm}^{\text{RSE1}} = \mathcal{K}_n^{\text{RSE1}} \cap \mathcal{K}_m^{\text{RSE1}}$ be the set of common sub-channels between user m and user n at the NSE and the RSE for Case 1, respectively. As per simulation results in Figs. 2-4, we have $|\mathcal{L}_{01}^{\text{RSE1}}| < |\mathcal{L}_{01}^{\text{NSE}}|$, where $|\mathcal{L}_{01}^{\text{RSE1}}|$ and $|\mathcal{L}_{01}^{\text{NSE}}|$ are the sizes of $\mathcal{L}_{01}^{\text{RSE1}}$ and $\mathcal{L}_{01}^{\text{NSE}}$, respectively. For example, in Scenario 1, we have $|\mathcal{L}_{01}^{\text{RSE1}}| = 17$ and $|\mathcal{L}_{01}^{\text{NSE}}| = 13$, and in Scenario 2, we have $|\mathcal{L}_{01}^{\text{RSE1}}| = 0$ and $|\mathcal{L}_{01}^{\text{NSE}}| = 1$. As expected and seen in Table IV, the leader's utility increases in Case 1 under all conditions. Interestingly, in Scenario 2, the follower's utility at the RSE is higher than that at the NSE. When the number of common sub-channels is reduced, there is less interference from the leader to the follower and vice versa at the RSE for Case 1 as compared to that at the NSE. This means that as shown in Table IV, it is possible that both the leader's utility and the follower's utility increase simultaneously. Fig. 5 shows the cumulative distribution function (CDF) of d_1^{RSE1} for the following examples of high interference between the leader and the follower. **Example 1:** the leader's interference to the follower is high, e.g., $\frac{h_{10}^k}{h_{11}^k} > 0.8$, and the follower's interference to the leader is low, e.g., $\frac{h_{01}^k}{h_{00}^k} < 0.1$; **Example 2:** the leader's and the follower's interference on each other are high, e.g., $\frac{h_{10}^k}{h_{11}^k} > 0.9$ and $\frac{h_{01}^k}{h_{00}^k} > 0.9$; and **Example 3:** the follower's interference to the leader is high, e.g., $\frac{h_{10}^k}{h_{11}^k} > 0.9$, and for the leader's interference on the follower is low, e.g., $\frac{h_{01}^k}{h_{00}^k} < 0.1$. Note that in Example 2, the follower's utility at the RSE is always below that at the NSE, whereas in Examples 1 and 3, with a probability of 10%, the follower's utility at the RSE may be higher than that at the NSE. In such instances, the leader and the follower may achieve higher utilities when there is uncertainty in the follower's observations with power constraint in (35), which is an interesting phenomenon in the RSG for the power control game.

VI. Extension to Multi-User Games

We now extend our analysis of the RSE for multi-leader multi-follower Stackelberg games, which is more challenging [30]–[32]. In doing so, we focuss on cases that the NE of followers' games is unique.

A. One-Leader Multi-Follower ($N_{\mathrm{L}} = 1$ and $N_{\mathrm{F}} > 1$)

For this scenario, there is only one leader in the Stackelberg game indexed by 0. Consider the $N_{\mathrm{F}} \times N_{\mathrm{F}}$ matrix $\boldsymbol{\Upsilon}$ whose elements are

$$[\Upsilon]_{nm} = \begin{cases} \alpha_n^{\min} & \text{if} \quad m = n, \quad m, n \in \mathcal{N}_{\mathrm{F}} \\ -\beta_{nm}^{\max} & \text{if} \quad m \neq n, \quad m, n \in \mathcal{N}_{\mathrm{F}}, \end{cases}$$

where

$$\alpha_n(\mathbf{a}) \triangleq \text{smallest eigenvalue of } -\nabla^2_{\mathbf{a}_n} v_n(\mathbf{a}_n, \mathbf{f}_n), \quad \alpha_n^{\min} \triangleq \inf_{\mathbf{a} \in \mathcal{A}} \alpha_n(\mathbf{a}), \quad \forall n \in \mathcal{N}_{\mathrm{F}} \tag{36}$$

$$\beta_{nm}(\mathbf{a}) \triangleq \| -\nabla_{\mathbf{a}_n \mathbf{a}_m} v_n(\mathbf{a}_n, \mathbf{f}_n)\|, \quad \forall n \neq m, \qquad \beta_{nm}^{\max} \triangleq \sup_{\mathbf{a} \in \mathcal{A}} \beta_n(\mathbf{a}), \quad \forall n \in \mathcal{N}_{\mathrm{F}}. \tag{37}$$

When $\boldsymbol{\Upsilon}$ is a P-matrix, the followers' NE is unique (Theorem 12.5 in [33]).

1) RSE for Case 1: In this case, the followers' observations are uncertain parameters modeled by (11). In contrast, the leader has complete information. For each follower, the optimization problem is similar to (13) and reformulations to (20) can be applied as

$$\widetilde{\mathbf{f}}_n^* = \mathbf{f}_n - \varepsilon_n \boldsymbol{\vartheta}_n, \qquad n \in \mathcal{N}_{\mathrm{F}}, \tag{38}$$

where $\widetilde{\mathbf{f}}_n^* = [\widetilde{f}_n^{1*}, \cdots, \widetilde{f}_n^{K*}]$, $\boldsymbol{\vartheta}_n = [\vartheta_n^1, \cdots, \vartheta_n^K]$, and

$$\vartheta_n^k = \frac{\frac{\partial u_n^k(\mathbf{a}_n, \widetilde{\mathbf{f}}_n)}{\partial f_n^k}}{\sqrt{\sum_{k=1}^K \left(\frac{\partial u_n^k(\mathbf{a}_n, \widetilde{\mathbf{f}}_n)}{\partial f_n^k}\right)^2}}. \tag{39}$$

Proposition 5. For the above Case 1 of the RSG, when $\boldsymbol{\Upsilon}$ is a P-matrix, we have: 1) the followers' strategies are decreasing functions of $\boldsymbol{\varepsilon} = [\varepsilon_1, \cdots, \varepsilon_N]$ and the social utility of the followers' game is less than that at the NSE; 2) the leader's utility at the RSE is higher than that at the NSE; and 3) the social utility at the RSE is higher than that at the NSE if

$$\text{C5}: \mathbf{J}^0_{\mathbf{a}_0} > \sum_{n\in\mathcal{N}_\text{F}} \mathbf{C}_{n0}\,, \qquad \text{C6}: \mathbf{J}^n_{\mathbf{a}_n} < \mathbf{C}_{0n} + \sum_{m\neq n, m\in\mathcal{N}_\text{F}} \mathbf{C}_{mn}, \quad \forall n \in \mathcal{N}_\text{F}.$$

Proof: See Appendix F. ■

Similar to Proposition 2, noise in the followers' observations increases the leader's utility, but reduces the followers' utilities. When the leader's direct rate is higher than its negative impact on the followers, i.e., C5, and when each follower's negative impacts on other followers and on the leader are greater than its direct rate, i.e., C6, the social utility at the RSE is higher than that at the NSE.

2) RSE for Case 2: In this case, we assume that $\mathbf{X}_{n0}$ is the uncertain parameter and the leader assumes that the uncertainty region for each follower is bounded to δ_{n0} described by

$$\mathcal{R}_{\mathbf{X}_{n0}} = \{\widetilde{\mathbf{X}}_{n0} \mid \|\widehat{\mathbf{X}}_{n0}\|_2 = \|\widetilde{\mathbf{X}}_{n0} - \mathbf{X}_{n0}\|_2 \leq \delta_{n0}\}, \quad \forall n \in \mathcal{N}_\text{F}. \tag{40}$$

Proposition 6. For Case 2, when $\boldsymbol{\Upsilon}$ is a P-matrix, we have: 1) The leader's utility at the RSE is always less than that at the NSE; 2) the followers' actions are increasing functions of $\boldsymbol{\delta}_0 = [\delta_{10}, \cdots, \delta_{N_f,0}]$, and the social utility of the followers' game at the RSE is higher than that at the NSE; and 3) the social utility increases if

$$\text{C7}: \mathbf{J}^0_{\mathbf{a}_0} < \sum_{n\in\mathcal{N}_\text{F}} \mathbf{C}_{n0}, \qquad \text{C8}: \mathbf{J}^n_{\mathbf{a}_n} > \mathbf{C}_{0n} + \sum_{m\neq n, m\in\mathcal{N}_\text{F}} \mathbf{C}_{mn}, \quad \forall n \in \mathcal{N}_\text{F}.$$

Proof: See Appendix G. ■

Again, the leader's incomplete information reduces its utility and increases the social utility of the followers. Also from C7-C8, when the leader's direct rate is less than its negative impact on the followers, and the negative impacts of each follower on other followers and on the leader are less than its direct rate, the social utility at the RSE for Case 2 increases as compared to that at the NSE. Besides, C5- C6, and C7-C8 are dual.

For the one-leader multi-follower scenario, the hierarchy between the leader and the followers still remains. To implement this scenario in a distributed manner, the leader announces its action first. Then, all the followers play their strategic non-cooperative robust game. The robust game between the followers can be implemented using a distributed algorithm as in [21].

B. *Multi-Leader Multi-Follower ($N_{\mathrm{L}} > 1$ and $N_{\mathrm{F}} > 1$)*

There are different forms of NSG for the multi-leader multi-follower scenario based on interactions among the leaders, i.e., cooperation or competition [2], [31]. Therefore, for this scenario, there are different forms of RSG and RSE. One possibility for multi-follower Stackelberg games is to consider cooperation between leaders when all leaders know each other and try to maximize their social utility via

$$\begin{aligned} &\max_{\mathbf{a}_n \in \mathcal{A}_n} \sum_{n \in \mathcal{N}_{\mathrm{L}}} \quad v_n(\mathbf{a}_n, \mathbf{f}_n) \\ &\text{subject to} \qquad \max_{\mathbf{a}_m \in \mathcal{A}_m} v_m(\mathbf{a}_m, \mathbf{f}_m), \qquad \forall m \in \mathcal{N}_{\mathrm{F}}. \end{aligned} \tag{41}$$

Proposition 7. When $\boldsymbol{\Upsilon}$ is a P-matrix for the followers' game, the leaders' social utility increases in Case 1 of the RSE.

Proof: See Appendix H. ■

In this scenario, the major concern is the analysis of Case 2. Since (41) is a non-convex and non-smooth optimization problem, deriving the conditions for increasing or decreasing the leaders' social utility (41) is impossible. Nevertheless, a heuristic protocol for increasing the social utility of followers and some of the leaders is proposed based on the results of this paper in Section IV. This protocol is summarized in Table V. Briefly, this protocol tries to find the leader that meets C7 and C8 among the leaders. If not found, the leader with the highest negative impact on the followers and on the other leaders is chosen. Then, the RSG for Case 2 with one-leader multi-follower in Section VI-A is played where the chosen leader plays first, and all other leaders and followers play after the chosen leader.

In Table VI, we evaluate the performance of heuristic algorithm where there are two leaders and only one follower in power control game. In SNE, both of the leaders maximize their sum of utility based on (41). When the information of leaders are subject to uncertainty, leader 1 acts as the leader and leader 2 as the follower. In this case, the uncertainty in the leaders causes the decreasing utilities of both of the leaders, while the follower's utility is increased. However, the social utility of the follower and the leader 2 is increased for the case 2 of RSE with heuristic algorithm. The above heuristic algorithm does not guarantee that the leader's social utility would be increased. However, it guarantees that social utility for all players excepts for one of the leaders is increased for Case 2 at the RSE.

VII. Conclusion

In this paper, we studied two cases for the robust Stackelberg equilibrium in communications systems: Case 1 where the leaders possess accurate information, while the followers' observations are noisy; and Case 2 where the leaders possess incomplete information sets, and the followers' observations are noisy.

We showed that the followers' noisy observations increase the leaders' utilities and reduce the followers' utilities; while the leaders' incomplete information increases the followers' utilities and reduces the leaders' utilities. We derived the conditions by which the social utility in these two cases is increased depending on the increase or the decrease in the leaders' and the followers' utilities. For power control games, we obtained the conditions for an increase in social utilities depending on channel gains in different SINR scenarios for these two cases. We also provided insights on how to increase the social utility in multi-leader scenarios with uncertain information in communications systems.

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APPENDIX A
PROOF OF LEMMA 1

Since $u_1(\mathbf{a}_1, \mathbf{f}_1)$ is a convex function of $\mathbf{f}_1$, the optimal value of $\widetilde{f}_1^k$ for the follower's optimization problem in (17) is obtained by

$$L(\mathbf{a}_1, \widetilde{\mathbf{f}}_1, \lambda) = \sum_{k=1}^{K} u_1^k(a_1^k, \widetilde{f}_1^k) + \lambda(\sum_{k=1}^{K}(\widetilde{f}_1^k - f_1^k)^2 - \varepsilon_1^2), \tag{42}$$

where λ is the nonnegative Lagrange multiplier that satisfies (11), i.e.,

$$\lambda \times (\varepsilon_1^2 - \sum_{k=1}^{K}(\widetilde{f}_1^k - f_1^k)^2) = 0. \tag{43}$$

From the optimality condition of the optimization problem (17), i.e., $\frac{\partial L(\mathbf{a}_1, \widetilde{\boldsymbol{f}}_1, \lambda)}{\partial \widetilde{f}_1^k} = 0$, we have

$$\frac{\partial u_1^k(a_1^k, \widetilde{f}_1^k)}{\partial \widetilde{f}_1^k} = -2\lambda \times (\widetilde{f}_1^k - f_1^k), \qquad \forall k \in \mathcal{K}. \tag{44}$$

By inserting the above solution into (43), the vector of uncertain variables is obtained as (19).

APPENDIX B
PROOF OF PROPOSITION 1

1) At the RSE for Case 1, the first order optimality condition is

$$\nabla_{\mathbf{a}_1^{*\mathrm{RSE1}}} u_1(\mathbf{a}_1^{*\mathrm{RSE1}}, \mathbf{f}_1^{*\mathrm{RSE1}}) = 0. \tag{45}$$

The derivative of (45) with respect to ε_1 is

$$[\nabla^2_{\mathbf{a}_1^{*\mathrm{RSE1}}} u_1(\mathbf{a}_1^{*\mathrm{RSE1}}, \mathbf{f}_1^{*\mathrm{RSE1}}) \nabla_{\varepsilon_1} \mathbf{a}_1^{*\mathrm{RSE1}} + \nabla_{\mathbf{a}_1^{*\mathrm{RSE1}} \mathbf{f}_1^{*\mathrm{RSE1}}} u_1(\mathbf{a}_1^{*\mathrm{RSE1}}, \mathbf{f}_1^{*\mathrm{RSE1}}) \nabla_{\varepsilon_1} \boldsymbol{f}_1^{*\mathrm{RSE1}}]_{\varepsilon_1=0} = 0. \tag{46}$$

Note that $\mathbf{f}_1^{*\mathrm{RSE1}}$ is a function of $\mathbf{a}_0^{*\mathrm{RSE1}}$ and $u_1(\mathbf{a}_1^{*\mathrm{RSE1}}, \mathbf{f}_1^{*\mathrm{RSE1}})|_{\varepsilon_1=0} = v_1(\mathbf{a}_1^{*\mathrm{NSE}}, \mathbf{f}_1^{*\mathrm{NSE}})$. Also, from (19), the last term on the left hand side of (46) is equal to $-\vartheta_1^{\mathrm{T}}$. By rearranging (46), we have

$$\nabla_{\varepsilon_1} \mathbf{a}_1^{*\mathrm{RSE1}} = (\mathbf{J}^1_{\mathbf{a}_1\mathbf{a}_1})^{-1} \mathbf{J}^1_{\mathbf{a}_1\mathbf{f}_1} \times \vartheta_1^{\mathrm{T}}, \tag{47}$$

where $\mathbf{J}^1_{\mathbf{a}_1\mathbf{a}_1} = \nabla^2_{\mathbf{a}_1^{*\mathrm{RSE1}}} u_1(\mathbf{a}_1^{*\mathrm{RSE1}}, \mathbf{f}_1^{*\mathrm{RSE1}})$ and $\mathbf{J}^1_{\mathbf{a}_1\mathbf{f}_1} = \nabla_{\mathbf{a}_1^{*\mathrm{RSE1}} \mathbf{f}_1^{*\mathrm{RSE1}}} u_1(\mathbf{a}_1^{*\mathrm{RSE1}}, \mathbf{f}_1^{*\mathrm{RSE1}})$. From A1 - A3, the right hand side of (47) is negative. Hence, $\nabla_{\varepsilon_1} \mathbf{a}_1^{*\mathrm{RSE1}} < 0$, meaning that the follower's action is a decreasing function of ε_1.

2) At the RSE for Case 1, the first order optimality condition for the utility of the leader is

$$\nabla_{\mathbf{a}_0^{*\mathrm{RSE1}}} v_0(\mathbf{a}_0^{*\mathrm{RSE1}}, \mathbf{f}_0^{*\mathrm{RSE1}}) = 0. \tag{48}$$

The derivative of (48) with respect to ε_1 is

$$[\nabla^2_{\mathbf{a}_0^{*\mathrm{RSE1}}} v_0(\mathbf{a}_0^{*\mathrm{RSE1}}, \mathbf{f}_0^{*\mathrm{RSE1}}) \times \nabla_{\varepsilon_1} \mathbf{a}_0^{*\mathrm{RSE1}} + \nabla_{\mathbf{a}_0^{*\mathrm{RSE1}} \mathbf{f}_0^{*\mathrm{RSE1}}} v_0(\mathbf{a}_0^{*\mathrm{RSE1}}, \mathbf{f}_0^{*\mathrm{RSE1}}) \times \mathbf{X}_{01} \times \nabla_{\varepsilon_1} \mathbf{a}_1^{*\mathrm{RSE1}}]_{\varepsilon_1=0} = 0, \tag{49}$$

which is equivalent to

$$\nabla_{\varepsilon_1} \mathbf{a}_0^{*\mathrm{RSE1}} = -(\mathbf{J}^0_{\mathbf{a}_0\mathbf{a}_0})^{-1} \mathbf{J}^0_{\mathbf{a}_0\mathbf{f}_0} \boldsymbol{X}_{01} \nabla_{\varepsilon_1} \mathbf{a}_1^{*\mathrm{RSE1}}, \tag{50}$$

where $\mathbf{J}^0_{\mathbf{a}_0\mathbf{a}_0} = \nabla^2_{\mathbf{a}_0^{*\mathrm{RSE1}}} v_0(\mathbf{a}_0^{*\mathrm{RSE1}}, \mathbf{f}_0^{*\mathrm{RSE1}})$ and $\mathbf{J}^0_{\mathbf{a}_0\mathbf{f}_0} = \nabla_{\mathbf{a}_0^{*\mathrm{RSE1}} \mathbf{f}_0^{*\mathrm{RSE1}}} v_0(\mathbf{a}_0^{*\mathrm{RSE1}}, \mathbf{f}_0^{*\mathrm{RSE1}})$. From A1 - A3, the right hand side of (50) is positive. Hence, $\nabla_{\varepsilon_1} \mathbf{a}_0^{*\mathrm{RSE1}} > 0$, meaning that the leaders' action is an increasing function of ε_1.

Appendix C
Proof of proposition 2

1) The Taylor series expansion of the leader's utility around the uncertain parameter is

$$v_0(\mathbf{a}_0^{*\mathrm{RSE1}}, \mathbf{f}_0^{*\mathrm{RSE1}}) = v_0(\mathbf{a}_0^{*\mathrm{NSE}}, \mathbf{f}_0^{*\mathrm{NSE}}) + \tag{51}$$
$$\varepsilon_1[(\mathbf{X}_{01} \nabla_{\mathbf{f}_0^{*\mathrm{RSE1}}} v_0(\mathbf{a}_0^{*\mathrm{RSE1}}, \mathbf{f}_0^{*\mathrm{RSE1}}))^{\mathrm{T}} \nabla_{\varepsilon_1} \mathbf{a}_1^{*\mathrm{RSE1}} + (\nabla_{\mathbf{a}_0^{*\mathrm{RSE1}}} v_0(\mathbf{a}_0^{*\mathrm{RSE1}}, \mathbf{f}_0^{*\mathrm{RSE1}})))^{\mathrm{T}} \nabla_{\varepsilon_1} \mathbf{a}_0^{*\mathrm{RSE1}}]_{\varepsilon_1=0} + o,$$

In the sequel, we only consider the first term of the Taylor series and ignore higher terms for small values of ε_1 because their values are very small (due to higher exponents of ε_1) compared to the first term.

From A2 and $\nabla_{\varepsilon_1} \mathbf{a}_1^{*\mathrm{RSE1}} < 0$, the second term on the right hand side of (51) is always positive. Also, the third term on the right hand side of (51) has positive elements only. Hence, the leader's utility is always greater than that at the NSE for Case 1. By some rearrangements, we have

$$\omega_0^{*\mathrm{RSE1}} - \omega_0^{*\mathrm{NSE}} \approx \varepsilon_1((\mathbf{J}^0_{\mathbf{a}_0})^{\mathrm{T}} \nabla_{\varepsilon_1} \mathbf{a}_0^{*\mathrm{RSE1}} + (\mathbf{X}_{10} \mathbf{J}^0_{\mathbf{f}_0})^{\mathrm{T}} \nabla_{\varepsilon_1} \mathbf{a}_1^{*\mathrm{RSE1}}). \tag{52}$$

2) From the Taylor series expansion of the follower's utility around ε_1, we have

$$u_1(\mathbf{a}_1^{*\mathrm{RSE1}}, \mathbf{f}_1^{*\mathrm{RSE1}}) = v_1(\mathbf{a}_1^{*\mathrm{NSE}}, \mathbf{f}_1^{*\mathrm{NSE}}) + \tag{53}$$
$$\varepsilon_1[(\mathbf{X}_{01} \nabla_{\mathbf{f}_1^{*\mathrm{RSE1}}} u_1(\mathbf{a}_1^{*\mathrm{RSE1}}, \mathbf{f}_1^{*\mathrm{RSE1}}))^{\mathrm{T}} \nabla_{\varepsilon_1} \mathbf{a}_0^{*\mathrm{RSE1}} + (\nabla_{\mathbf{a}_1^{*\mathrm{RSE1}}} u_1(\mathbf{a}_1^{*\mathrm{RSE1}}, \mathbf{f}_1^{*\mathrm{RSE1}}))^{\mathrm{T}} \times \nabla_{\varepsilon_1} \mathbf{a}_1^{*\mathrm{RSE1}}]_{\varepsilon_1=0} + o,$$

Since $\nabla_{\mathbf{f}_1} v_1(\mathbf{a}_1, \mathbf{f}_1) < 0$ and $\nabla_{\varepsilon_1} \mathbf{a}_0^{*\mathrm{RSE1}} > 0$, the second term on the right hand side of (53) is always negative. Also, $\nabla_{\mathbf{a}_1} v_1(\mathbf{a}_1, \mathbf{f}_1) > 0$ and $\nabla_{\varepsilon_1} \mathbf{a}_1^{*\mathrm{RSE1}} < 0$. Consequently, the third term on the right hand side of (53) is negative. Hence, the follower's utility at the RSE for Case 1 is always less than its utility at the NSE. By some rearrangements, we have

$$\omega_1^{*\mathrm{RSE1}} - \omega_1^{*\mathrm{NSE}} \approx \varepsilon_1 \times ((\mathbf{X}_{01} \mathbf{J}^1_{\mathbf{f}_1})^{\mathrm{T}} \nabla_{\varepsilon_1} \mathbf{a}_0^{*\mathrm{RSE1}} + (\mathbf{J}^1_{\mathbf{a}_1})^{\mathrm{T}} \nabla_{\varepsilon_1} \mathbf{a}_1^{*\mathrm{RSE1}}). \tag{54}$$

3) The social utility at the RSE for Case 1 is increased when

$$\omega_0^{*\text{RSE1}} - \omega_0^{*\text{NSE}} + \omega_1^{*\text{RSE1}} - \omega_1^{*\text{NSE}} > 0, \tag{55}$$

which is equivalent to the sum of (52) and (54). Since $\nabla_{\varepsilon_1}\mathbf{a}_1^{*\text{RSE1}} < 0$ and $\nabla_{\varepsilon_1}\mathbf{a}_0^{*\text{RSE1}} > 0$, the sum of the terms multiplied by $\nabla_{\varepsilon_1}\mathbf{a}_1^{*\text{RSE1}}$ is negative and the sum of the terms multiplied by $\nabla_{\varepsilon_1}\mathbf{a}_0^{*\text{RSE1}}$ is positive. Hence, we have

$$|\mathbf{J}_{\mathbf{a}_0}^0| - |\mathbf{X}_{01}||\mathbf{J}_{\mathbf{f}_1}^1| > 0, \qquad |\mathbf{J}_{\mathbf{a}_1}^1| - |\mathbf{X}_{10}||\mathbf{J}_{\mathbf{f}_0}^0| < 0, \tag{56}$$

which are equal to C1 and C2.

Appendix D
Proof of Proposition 3

The leader cannot calculate the exact value of $\mathbf{a}_1$ from its incomplete information set, meaning that the value of $\mathbf{f}_0$ is uncertain. Consequently, the RSE for Case 2 can be considered as ϵ Stackelberg strategy space for Case 1 (Definition 4.7 in [1]). Since the leader's utility is continuous (from A1 - A2), there always exists a positive value $\varsigma > 0$, for which $\omega_0^{*\text{RSE1}} - \omega_0^{*\text{RSE2}} \leq \varsigma$ (Property 4.2 in [1]) and this difference is continuous (Property 4.3 in [1]).

Appendix E
Proof of Proposition 4

1) To obtain $\omega_0^{*\text{RSE2}} - \omega_0^{*\text{NSE}}$, we need to derive $\nabla_{\mathbf{f}_1^{*\text{RSE2}}}\mathbf{a}_1^{*\text{RSE2}}$. At the RSE for Case 2, we have

$$\nabla_{\mathbf{a}_1^{*\text{RSE2}}} u_1(\mathbf{a}_1^{*\text{RSE2}}, \mathbf{f}_1^{*\text{RSE2}}) = 0. \tag{57}$$

The derivative of (57) with respect to $\mathbf{f}_1^{*\text{RSE2}}$ is

$$\mathbf{J}_{\mathbf{a}_1\mathbf{a}_1}^1 \nabla_{\mathbf{f}_1^{*\text{RSE2}}}\mathbf{a}_1^{*\text{RSE2}} + \mathbf{J}_{\mathbf{a}_1\mathbf{f}_1}^1 = 0, \tag{58}$$

which means $\nabla_{\mathbf{f}_1^{*\text{RSE2}}}\mathbf{a}_1^{*\text{RSE2}} = -(\mathbf{J}_{\mathbf{a}_1\mathbf{a}_1}^1)^{-1}\mathbf{J}_{\mathbf{a}_1\mathbf{f}_1}^1$ is always negative. On the other hand, $\nabla_{\delta_{10}}\mathbf{a}_1^{*\text{RSE2}}$ is

$$\nabla_{\delta_{10}}\mathbf{a}_1^{*\text{RSE2}} = \nabla_{\mathbf{f}_1^{*\text{RSE2}}}\mathbf{a}_1^{*\text{RSE2}} \nabla_{\delta_{10}}\mathbf{f}_1^{*\text{RSE2}}, \tag{59}$$

and from A5, (59) is always positive, meaning that the follower's action is an increasing function of δ_{10}. The Taylor series expansion of $u_1(\mathbf{a}^{*\text{RSE2}}, \mathbf{f}_1^{*\text{RSE2}})$ around δ_{10} is

$$u_1(\mathbf{a}^{*\text{RSE2}}, \mathbf{f}_1^{*\text{RSE2}}) = v_1(\mathbf{a}^{*\text{NSE}}, \mathbf{f}_1^{*\text{NSE}}) + \delta_{10} \times [\frac{\partial u_1(\mathbf{a}_1^{*\text{RSE2}}, \mathbf{f}_1^{*\text{RSE2}})}{\partial \delta_{10}}]_{\delta_{10}=0,\varepsilon_1=0} + o, \tag{60}$$

where

$$\frac{\partial u_1(\mathbf{a}_1^{*\text{RSE2}},\mathbf{f}_1^{*\text{RSE2}})}{\partial \delta_{10}} = $$
$$(\nabla_{\mathbf{a}_1^{*\text{RSE2}}} u_1(\mathbf{a}_1^{*\text{RSE2}},\mathbf{f}_1^{*\text{RSE2}}))^{\text{T}} \nabla_{\mathbf{f}_1^{*\text{RSE2}}} \mathbf{a}_1^{*\text{RSE2}} \nabla_{\delta_{10}} \mathbf{f}_1^{*\text{RSE2}} +$$
$$(\nabla_{\mathbf{f}_1^{*\text{RSE2}}} u_1(\mathbf{a}_1^{*\text{RSE2}},\mathbf{f}_1^{*\text{RSE2}}))^{\text{T}} \nabla_{\delta_{10}} \mathbf{f}_1^{*\text{RSE2}}. \tag{61}$$

From (58) and (59), (61) is equivalent to

$$\omega_1^{*\text{RSE2}} - \omega_1^{*\text{NSE}} \approx \delta_{10} \times ((\mathbf{J}^1_{\mathbf{a}_1})^{\text{T}} \mathbf{J}^1_{\mathbf{a}_1\mathbf{f}_1} (\mathbf{J}^1_{\mathbf{a}_1\mathbf{a}_1})^{-1} - (\mathbf{J}^1_{\mathbf{f}_1})^{\text{T}}) \nabla_{\delta_{10}} \mathbf{f}_1^{*\text{RSE2}}. \tag{62}$$

From A1 - A5, the right hand side of (62) is positive. Therefore, the follower's utility at the RSE for Case 2 is higher than that at the NSE.

2) From $\mathbf{J}^0_{\mathbf{a}_0} = 0$, we have

$$\mathbf{J}^0_{\mathbf{a}_0\mathbf{a}_0} \nabla_{\delta_{10}} \mathbf{a}_0^{*\text{RSE2}} + \mathbf{X}_{01} \mathbf{J}^0_{\mathbf{a}_0\mathbf{f}_0} \nabla_{\delta_{10}} \mathbf{a}_1 = 0, \tag{63}$$

which is equivalent to $\nabla_{\delta_{10}} \mathbf{a}_0^{*\text{RSE2}} = -(\mathbf{J}^0_{\mathbf{a}_0\mathbf{a}_0})^{\text{T}} \mathbf{X}_{01} \mathbf{J}^0_{\mathbf{a}_0\mathbf{f}_0} \nabla_{\delta_{10}} \mathbf{a}_1$ and is always negative from A1 - A5 and (59). Hence, the leader's strategy is a decreasing function of δ_{10}. For the leader's utility function, we start with $\mathbf{J}^0_{\mathbf{a}_0} = 0$ to derive $\nabla_{\mathbf{a}_1^{*\text{RSE2}}} \mathbf{a}_0^{*\text{RSE2}}$. The derivative of $\mathbf{J}^0_{\mathbf{a}_0} = 0$ at $\mathbf{a}_1^{*\text{RSE2}}$ is

$$\mathbf{J}^0_{\mathbf{a}_0\mathbf{a}_0} \nabla_{\mathbf{a}_1^{*\text{RSE2}}} \mathbf{a}_0^{*\text{RSE2}} + \mathbf{X}_{01} \mathbf{J}^0_{\mathbf{a}_0\mathbf{f}_0} = 0, \tag{64}$$

From the Taylor series expansion of $v_0(\mathbf{a}_0^{*\text{RSE2}},\mathbf{f}_0^{*\text{RSE2}})$ around δ_{10}, we have

$$v_0(\mathbf{a}_0^{*\text{RSE2}},\mathbf{f}_0^{*\text{RSE2}}) = v_0(\mathbf{a}_0^{*\text{NSE}},\mathbf{f}_0^{*\text{NSE}}) + \delta_{10} \times [\frac{\partial v_0(\mathbf{a}_0^{*\text{RSE2}},\mathbf{f}_0^{*\text{RSE2}})}{\partial \delta_{10}}]_{\delta_{10}=0} + o, \tag{65}$$

where

$$\frac{\partial v_0(\mathbf{a}_0^{*\text{RSE2}},\mathbf{f}_0^{*\text{RSE2}})}{\partial \delta_{10}} = $$
$$(\nabla_{\mathbf{a}_0^{*\text{RSE2}}} v_0(\mathbf{a}_0^{*\text{RSE2}},\mathbf{f}_0^{*\text{RSE2}}))^{\text{T}} \nabla_{\mathbf{a}_1^{*\text{RSE2}}} \mathbf{a}_0^{*\text{RSE2}} \nabla_{\delta_{10}} \mathbf{a}_1^{*\text{RSE2}} +$$
$$(\nabla_{\mathbf{f}_0} v_0(\mathbf{a}_0^{*\text{RSE2}},\mathbf{f}_0^{*\text{RSE2}}))^{\text{T}} \mathbf{X}_{01} \nabla_{\delta_{10}} \mathbf{a}_1^{*\text{RSE2}} + o, \tag{66}$$

By inserting (64) in (66), we have

$$\omega_0^{*\text{RSE2}} - \omega_0^{*\text{NSE}} \approx \delta_{10} \times [-(\mathbf{J}^0_{\mathbf{a}_0})^{\text{T}} \mathbf{J}^0_{\mathbf{a}_0\mathbf{f}_0} \mathbf{X}_{01} (\mathbf{J}^0_{\mathbf{a}_0\mathbf{a}_0})^{-1} + (\mathbf{J}^0_{\mathbf{f}_0})^{\text{T}} \mathbf{X}_{01}] \nabla_{\delta_{10}} \mathbf{a}_1, \tag{67}$$

which, from A1 - A5, is always negative.

3) Now we derive the conditions for increasing the social utility. Since $\nabla_{\mathbf{a}_1^{*\text{RSE2}}} \mathbf{a}_0^{*\text{RSE2}} \times \nabla_{\delta_{10}} \mathbf{a}_0^{*\text{RSE2}} < 0$, the sum of the second term on the right hand side of (62) and the first term on the right hand side of (67) is negative, i.e.,

$$|\mathbf{J}^0_{\mathbf{a}_0}| - |\mathbf{J}^1_{\mathbf{f}_1}||\mathbf{X}_{10}| < 0, \tag{68}$$

Since $\nabla_{\delta_{10}}\mathbf{a}_1^{*\mathrm{RSE2}} > 0$, the sum of the first term on the right hand side of (62) and the second term on the right hand side of (67) is positive, i.e.,

$$|\mathbf{J}_{\mathbf{a}_1}^1| - |\mathbf{J}_{\mathbf{f}_0}^0||\mathbf{X}_{01}| > 0. \tag{69}$$

Clearly, (69) and (68) are equivalent to C3 and C4.

Appendix F
Proof of proposition 5

Lemma 2. When $\boldsymbol{\Upsilon}$ is a P-matrix, the followers' strategies are decreasing functions of $\boldsymbol{\varepsilon} = [\varepsilon_1, \cdots, \varepsilon_{N_\mathrm{F}}]$.

Proof: Consider $\mathbf{a}_{\mathrm{N_F}} = [\mathbf{a}_1, \cdots, \mathbf{a}_{N_\mathrm{F}}]$ and assume that $\mathbf{a}_{\mathrm{N_F}}$ is an increasing function of $\boldsymbol{\varepsilon}$, i.e.,

$$\mathbf{a}_{\mathrm{N_F}}^{*\mathrm{RSE1}} \geq \mathbf{a}_{\mathrm{N_F}}^{*\mathrm{NSE}}, \tag{70}$$

When $\boldsymbol{\Upsilon}$ is a P-matrix, $\boldsymbol{\mathcal{J}}(\mathbf{a}_{\mathrm{N_F}}) = (\mathbf{J}_{\mathbf{a}_n}^n(\mathbf{a}_n))_{n=1}^{N_\mathrm{F}}$ is strictly monotone (Theorem 12.5 in [33]), and we have

$$\boldsymbol{\mathcal{J}}(\mathbf{a}_{\mathrm{N_F}}^{*\mathrm{RSE1}}) \geq \boldsymbol{\mathcal{J}}(\mathbf{a}_{\mathrm{N_F}}^{*\mathrm{NSE}}). \tag{71}$$

On the other hand, from (38), we have

$$\frac{\partial u_n^k(a_n^k, f_n^k)}{\partial a_n^k} = \frac{\partial v_n^k(a_n^k, \widetilde{f}_n^{k*})}{\partial a_n^k} + \frac{\partial v_n^k(a_n^k, \widetilde{f}_n^{k*})}{\partial \widetilde{f}_n^{k*}} \times \frac{\partial \widetilde{f}_n^{k*}}{\partial a_n^k}, \tag{72}$$

and

$$\frac{\partial \widetilde{f}_n^{k*}}{\partial a_n^k} = \frac{\partial \widetilde{f}_n^{k*}}{\partial \vartheta_n^k} \times \frac{\partial \vartheta_n^k}{\partial a_n^k} = -\varepsilon_1 \times \frac{\partial^2 v_n^k(\mathbf{a}_n, \widetilde{f}_n^*)}{\partial a_n^k \partial \widetilde{f}_n^k} \times (\sum_{k=1}^{K} (\frac{\partial u_n^k(\mathbf{a}_n, \widetilde{\mathbf{f}}_n^*)}{\partial f_n^k})^2)^{-\frac{1}{2}}. \tag{73}$$

Consider $\widetilde{a}_n^k = -\varepsilon_1 \times \frac{\partial v_n^k(\mathbf{a}_n, \widetilde{\mathbf{f}}_n^*)}{\partial \widetilde{f}_n^{k*}} \times \frac{\partial^2 v_n^k(\mathbf{a}_n, \widetilde{\mathbf{f}}_n^*)}{\partial a_n^k \partial \widetilde{f}_n^{k*}} \times (\sum_{k=1}^{K} (\frac{\partial u_n^k(\mathbf{a}_n, \widetilde{\mathbf{f}}_n^*)}{\partial f_n^{k*}})^2)^{-\frac{1}{2}}|_{\mathbf{a}_n = \mathbf{a}_n^{\mathrm{NSE}}}$, which is negative according to A1 - A3. We rewrite (73) as

$$\boldsymbol{\mathcal{J}}(\mathbf{a}_{\mathrm{N_F}}^{*\mathrm{RSE1}}) - \boldsymbol{\mathcal{J}}(\mathbf{a}_{\mathrm{N_F}}^{*\mathrm{NSE}}) = \widetilde{\mathbf{a}} < \mathbf{0}, \tag{74}$$

where $\widetilde{\mathbf{a}} = (\widetilde{\mathbf{a}}_n)_{n=1}^{N_\mathrm{F}}$, $\widetilde{\mathbf{a}}_n^{\mathrm{T}} = [\widetilde{a}_n^1, \cdots, \widetilde{a}_n^K]$ and $\mathbf{0}$ is the zero vector whose size is the same as $\widetilde{\mathbf{a}}$. Obviously, (74) contradicts (71), which implies that our assumption was wrong. Consequently, the followers' actions at the RSE for Case 1 are decreasing functions of $\boldsymbol{\varepsilon}$. ∎

1) Since the followers' strategies are decreasing functions of $\boldsymbol{\varepsilon}$, the value of $\mathbf{f}_0$ is reduced with increasing $\boldsymbol{\varepsilon}$, which implies $v_0^{\mathrm{RSE1}} \geq v_0^{\mathrm{NSE}}$ from A2. Besides, the Taylor series expansion of v_0^{RSE1} around $\boldsymbol{\varepsilon}$ is

$$\omega_0^{*\mathrm{RSE1}} \approx \omega_0^{*\mathrm{NSE}} + [(\nabla_{\boldsymbol{\varepsilon}}\mathbf{a}_0)^{\mathrm{T}}\mathbf{J}_{\mathbf{a}_0}^n + \sum_{n=1}^{N_\mathrm{F}} \varepsilon_n \times [\mathbf{X}_{0n}\mathbf{J}_{\mathbf{f}_0}^0(\nabla_{\varepsilon_n}\mathbf{a}_n)^{\mathrm{T}}] + o. \tag{75}$$

2) The RNE of the followers in the multi-follower RSG in Section VI. A. belongs to the robust additively coupled games introduced in [21]. Based on Theorem 2 in [21], when $\boldsymbol{\Upsilon}$ is a P-matrix, the followers' social utility at the RSE is less than that at the NSG. Also, the Taylor series expansion of the utility of n^{th} follower around $\boldsymbol{\varepsilon}$ is

$$\omega_n^{*\text{RSE1}} \approx \omega_n^{*\text{NSE}} + \varepsilon_n \times [(\mathbf{J}_{\mathbf{f}_n}^n)^{\text{T}} \mathbf{X}_{n0} \nabla_{\boldsymbol{\varepsilon}} \mathbf{a}_0 + \sum_{m=1, m\neq n}^{N_{\text{F}}} \mathbf{X}_{nm} \nabla_{\varepsilon_m} \mathbf{a}_m + (\mathbf{J}_{\mathbf{a}_n}^n)^{\text{T}} \nabla_{\varepsilon_n} \mathbf{a}_n] + o. \tag{76}$$

3) When the sum of (75) and (76) is positive, the social utility of the RSG at the RSE for Case 1 is higher than that at the NSE. Hence, the terms multiplied by $\nabla_{\varepsilon_n} \mathbf{a}_0$ are positive because $\nabla_{\boldsymbol{\varepsilon}} \mathbf{a}_0 > 0$. Since $\nabla_{\varepsilon_n} \mathbf{a}_n < 0$, its multiplied terms are negative. By some rearrangements, positiveness of multiplicants to $\nabla_{\boldsymbol{\varepsilon}} \mathbf{a}_0$ and negativeness of multiplicants to $\nabla_{\varepsilon_n} \mathbf{a}_n$ lead to C5 and C6.

APPENDIX G
PROOF OF PROPOSITION 6

Lemma 3. When $\boldsymbol{\Upsilon}$ is a P-matrix, the followers' strategies are increasing functions of $\boldsymbol{\delta}_0 = [\delta_{10}, \cdots, \delta_{N_{\text{F}}0}]$.

Proof: We assume that the follower's strategies $\mathbf{a}_{\text{N}_{\text{F}}} = [\mathbf{a}_1, \cdots, \mathbf{a}_{N_{\text{F}}}]$ are decreasing functions of $\boldsymbol{\delta}_0$, i.e., $\mathbf{a}_{\text{N}_{\text{F}}}^{*\text{RSE2}} \leq \mathbf{a}_{\text{N}_{\text{F}}}^{*\text{NSE}}$. When $\boldsymbol{\Upsilon}$ is a P-matrix, $\boldsymbol{\mathcal{J}}(\mathbf{a}_{\text{N}_{\text{F}}}) = (\mathbf{J}_{\mathbf{a}_n}^n(\mathbf{a}_n))_{n=1}^{N_{\text{F}}}$ is strictly monotone (Theorem 12.5 in [33]), and we have

$$\boldsymbol{\mathcal{J}}(\mathbf{a}_{\text{N}_{\text{F}}}^{*\text{RSE2}}) \leq \boldsymbol{\mathcal{J}}(\mathbf{a}_{\text{N}_{\text{F}}}^{*\text{NSE}}) \tag{77}$$

From the Taylor series expansion of the followers' utilities around $\boldsymbol{\delta}_0$, we have

$$\frac{\partial u_n^k(\mathbf{a}_n, \mathbf{f}_n^*)}{\partial a_n^k} = \frac{\partial v_n^k(a_n^k, \widetilde{f}_n^{k*})}{\partial a_n^k} + \frac{\partial v_n^k(a_n^k, \widetilde{f}_n^{k*})}{\partial \widetilde{f}_n^{k*}} \times \frac{\partial \widetilde{f}_n^{k*}}{\partial \delta_{n0}}. \tag{78}$$

From A5, the last term in (78) is positive and

$$\boldsymbol{\mathcal{J}}(\mathbf{a}_{\text{N}_{\text{F}}}^{*\text{RSE2}}) - \boldsymbol{\mathcal{J}}(\mathbf{a}_{\text{N}_{\text{F}}}^{*\text{NSE}}) = \widetilde{\mathbf{b}} > \mathbf{0} \tag{79}$$

where $\widetilde{\mathbf{b}} = (\widetilde{\mathbf{b}}_n)_{n=1}^{N_{\text{F}}}$ and the k^{th} element of $\widetilde{\mathbf{b}}_n$ is equal to $\frac{\partial v_n^k(a_n^k, \widetilde{f}_n^{k*})}{\partial \widetilde{f}_n^{k*}} \times \frac{\partial \widetilde{f}_n^{k*}}{\partial \delta_{n0}}|_{\mathbf{a}=\mathbf{a}^{*\text{NSE}}}$. From (79), we have $\boldsymbol{\mathcal{J}}(\mathbf{a}_{\text{N}_{\text{F}}}^{*\text{RSE2}}) > \boldsymbol{\mathcal{J}}(\mathbf{a}_{\text{N}_{\text{F}}}^{*\text{NSE}})$, which contradicts (77). Hence, the followers' strategies are increasing functions of $\boldsymbol{\delta}_0$. ■

1) When the followers' actions are increasing functions of $\boldsymbol{\delta}_0$, the value of $\mathbf{f}_0$ increases. Since v_0 is a decreasing function of $\mathbf{f}_0$, the leader's utility is a decreasing function of $\boldsymbol{\delta}_0$. Form the Taylor series expansion of the leader's utility around $\boldsymbol{\delta}_\mathbf{0}$, we have

$$\omega_0^{*\text{RSE2}} \approx \omega_0^{*\text{NSE}} + \sum_{n\in\mathcal{N}_{\text{F}}} \delta_{n0} \times ((\mathbf{J}_{\mathbf{f}_0}^0)^{\text{T}} \mathbf{X}_{0n} \nabla_{\delta_{n0}} \mathbf{a}_n + (\mathbf{J}_{\mathbf{a}_0}^0)^{\text{T}} \nabla_{\delta_{n0}} \mathbf{a}_0) + o. \tag{80}$$

TABLE I

SUMMARY OF PROPOSITIONS 2 AND 4.

Case of robust game	Leader	Follower	The Social Utility Increases If
Case 1	$\omega_0^{\text{*RSE1}} \geq \omega_0^{\text{*NSE}}$	$\omega_1^{\text{*RSE1}} \leq \omega_1^{\text{*NSE}}$	$\lvert \mathbf{J}_{\mathbf{a}_0}^0 \rvert > \lvert \mathbf{C}_{10} \rvert$, $\lvert \mathbf{J}_{\mathbf{a}_1}^1 \rvert < \lvert \mathbf{C}_{01} \rvert$
Case 2	$\omega_0^{\text{*RSE2}} \leq \omega_0^{\text{*NSE}}$	$\omega_1^{\text{*RSE2}} \geq \omega_1^{\text{*NSE}}$	$\lvert \mathbf{J}_{\mathbf{a}_0}^0 \rvert < \lvert \mathbf{C}_{10} \rvert$, $\lvert \mathbf{J}_{\mathbf{a}_1}^1 \rvert > \lvert \mathbf{C}_{01} \rvert$

2) Assume that the followers' utilities are decreasing functions of $\boldsymbol{\delta}_0$. In this case, the followers' strategies are decreasing functions of $\boldsymbol{\delta}_0$. This is because when $\boldsymbol{\Upsilon}$ is a P-matrix, $\mathbf{J}_{\mathbf{a}}^n$ is strong monotone. However, this contradicts Lemma 3, meaning that the followers' utilities are increasing functions of $\boldsymbol{\delta}_0$. Consequently, the social utility of the followers' game is higher than that at the NSE. Besides, the Taylor series expansion of the n^{th} follower's utility around δ_{n0} is

$$\omega_n^{\text{*RSE2}} \approx \omega_n^{\text{*NSE}} + \delta_{n0} \times [(\mathbf{J}_{\mathbf{f}_n}^n)^{\text{T}} \mathbf{X}_{n0} \nabla_{\delta_{0n}} \mathbf{a}_0 + (\mathbf{J}_{\mathbf{f}_n}^n)^{\text{T}} (\sum_{m \neq n,\, m \in \mathcal{N}_{\text{F}}} \mathbf{X}_{nm} \nabla_{\delta_{m0}} \mathbf{a}_m) + (\mathbf{J}_{\mathbf{a}_n}^n)^{\text{T}} \nabla_{\delta_{0n}} \mathbf{a}_n] + o. \quad (81)$$

3) When the sum of the second term in (80) and (81) for all followers are positive, the social utility at the RSE for Case 2 is higher than that at the NSE. In this case, $\nabla_{\delta_{n0}} \mathbf{a}_0 < 0$ and $\nabla_{\delta_{n0}} \mathbf{a}_n > 0$. Hence, the terms multiplied by $\nabla_{\boldsymbol{\delta}_0} \mathbf{a}_0$ are negative, and the terms multiplied by $\nabla_{\delta_{n0}} \mathbf{a}_n$ are positive. By some rearrangements, C7 and C8 are obtained.

APPENDIX H
PROOF OF PROPOSITION 7

The RSE for the followers in the multi-leader multi-follower RSG in Section VI-B belongs to the robust additively coupled games introduced in [21]. The followers' strategies at the RSE for Case 1 are decreasing functions of the size of their uncertainty regions (Theorem 2 in [21]). Consequently, introducing robustness reduces the followers' impacts on the leaders, and increases the leaders' utilities, meaning that the leaders' social utility at the RSE for case 1 is higher than that at the NSE.

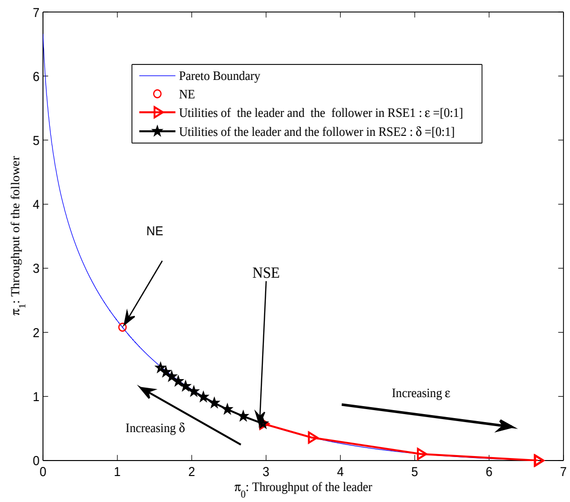

Fig. 1. Variations in the utilities of the follower and the leader for Cases 1 and 2 of the RSG.

TABLE II

VALIDATING PROPOSITION 2 VIA A POWER CONTROL NUMERICAL EXAMPLE

$R1$		$\varepsilon = 0$	$\varepsilon = 20\%$	$\varepsilon = 40\%$	$\varepsilon = 60\%$	$\varepsilon = 80\%$	$\varepsilon = 100\%$
(29)-(31) hold	d_0^{RSE1}	0	1.57	3.21	4.92	6.7	8.56
	d_1^{RSE1}	0	-1.7	-3.5	-5.33	-7.3	-9.2
	d^{RSE1}	0	0.02	0.03	0.037	0.045	0.045
(29)-(31) do not hold	d_0^{RSE1}	0	2.3	4.8	7.4	10.2	13.2
	d_1^{RSE1}	0	-2.14	-4.38	-6.75	-9.24	-11.9
	d^{RSE1}	0	-0.11	-0.21	-0.31	-0.4	-0.47
$R2$		$\varepsilon = 0$	$\varepsilon = 20\%$	$\varepsilon = 40\%$	$\varepsilon = 60\%$	$\varepsilon = 80\%$	$\varepsilon = 100\%$
(29)-(31) hold	d_0^{RSE1}	0	33.5	138.7	224.81	224.81	224.81
	d_1^{RSE1}	0	-35.6	-87.8	-100	-100	-100
	d^{RSE1}	0	7.1	52.15	100.71	100.71	100.71
(29)-(31) do not hold	d_0^{RSE1}	0	5.7	12.2	19.7	28.38	38.6
	d_1^{RSE1}	0	-4.27	-8.85	-13.82	-19.22	-25.13
	d^{RSE1}	0	-0.66	-1.24	-1.72	-2.04	-2.13
$R3$		$\varepsilon = 0$	$\varepsilon = 20\%$	$\varepsilon = 40\%$	$\varepsilon = 60\%$	$\varepsilon = 80\%$	$\varepsilon = 100\%$
(29)-(31) hold	d_0^{RSE1}	0	101.1	101.26	101.26	101.26	101.26
	d_1^{RSE1}	0	-100	-100	-100	-100	-100
	d^{RSE1}	0	53.7	53.8	53.8	53.8	53.8
(29)-(31) do not hold	d_0^{RSE1}	0	0.07	0.14	0.20	0.25	0.28
	d_1^{RSE1}	0	-0.06	-0.12	-0.17	-0.22	-0.26
	d^{RSE1}	0	-0.04	-0.07	-0.1	-0.13	-0.16

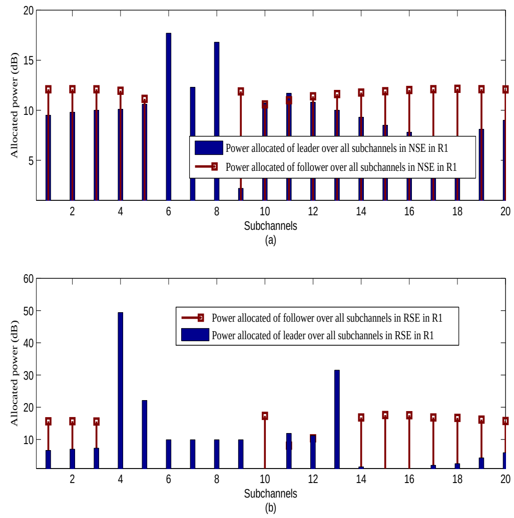

Fig. 2. Power allocation to the follower and to the leader in Scenario 1 subject to (35) - (a): at NSE, and (b): at RSE for Case 1.

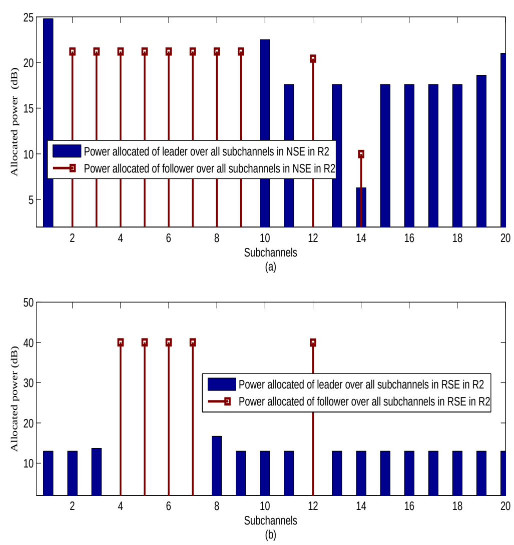

Fig. 3. Power allocation to the follower and to the leader in Scenario 2 subject to (35) - (a): at NSE and (b): at RSE for Case 1.

TABLE III

VALIDATING PROPOSITION 4 VIA A POWER CONTROL NUMERICAL EXAMPLE

$R1$		$\delta_{10}=0$	$\delta_{10}=20\%$	$\delta_{10}=40\%$	$\delta_{10}=60\%$	$\delta_{10}=80\%$	$\delta_{10}=100\%$
(32)-(34) do not hold	d_0^{RSE1}	0	-0.31	-0.62	-0.92	-1.23	-1.53
	d_1^{RSE1}	0	0.31	0.62	0.93	1.23	1.53
	d^{RSE1}	0	-0.01	-0.02	-0.03	-0.04	-0.05
(32)-(34) hold	d_0^{RSE1}	0	-0.23	-0.45	-0.67	-0.89	-1.12
	d_1^{RSE1}	0	0.18	0.37	0.55	0.74	0.92
	d^{RSE1}	0	0.002	0.004	0.005	0.007	0.008
$R2$		$\delta_{10}=0$	$\delta_{10}=20\%$	$\delta_{10}=40\%$	$\delta_{10}=60\%$	$\delta_{10}=80\%$	$\delta_{10}=100\%$
(32)-(34) do not hold	d_0^{RSE1}	0	-6.65	-12.37	-17.4	-21.76	-25.7
	d_1^{RSE1}	0	8.7	16.7	24.24	31.24	37.8
	d^{RSE1}	0	-0.08	-1.25	-1.42	-1.47	-1.51
(32)-(34) hold	d_0^{RSE1}	0	-3.86	-7.43	-10.72	-13.78	-16.63
	d_1^{RSE1}	0	2.94	5.74	8.4	10.99	13.46
	d^{RSE1}	0	0.49	0.99	1.51	2.05	2.59
$R3$		$\delta_{10}=0$	$\delta_{10}=20\%$	$\delta_{10}=40\%$	$\delta_{10}=60\%$	$\delta_{10}=80\%$	$\delta_{10}=100\%$
(32)-(34) do not hold	d_0^{RSE1}	0	-5.86	-10.86	-15.2	-19.06	-22.49
	d_1^{RSE1}	0	11.99	23.01	33.2	42.7	51.5
	d^{RSE1}	0	-1.65	-2.88	-3.81	-4.51	-5.05
(32)-(34) hold	d_0^{RSE1}	0	-1.47	-2.91	-4.3	-5.65	-6.96
	d_1^{RSE1}	0	0.65	1.29	1.91	2.53	3.13
	d^{RSE1}	0	0.24	0.47	0.69	0.93	1.157

TABLE IV

THE LEADER'S AND THE FOLLOWER'S UTILITIES SUBJECT TO (35).

Different Scenarios Based on SINR	R1			R2			R3		
Achieved Utility for Case 1	RSE	NSE	$d\%$	RSE	NSE	$d\%$	RSE	NSE	$d\%$
Leader	92.17	71.95	28.1%	110.11	105.71	4.2%	156.01	119.89	30.13%
Follower	53.47	70.84	-24.5%	97.67	95.94	1.8%	50.15	96.55	-48.06%

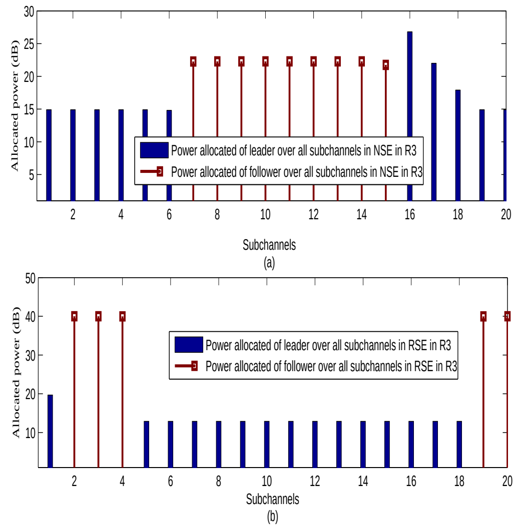

Fig. 4. Power allocation to the follower and to the leader in Scenario 2 subject to (35) - (a): at NSE and (b): at RSE for Case 1.

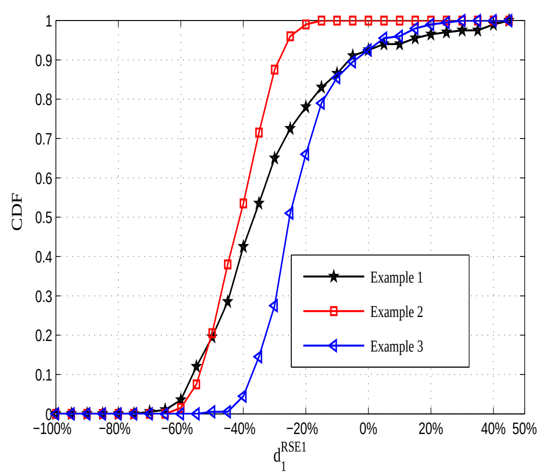

Fig. 5. CDF of d_1^{RSE1} when (35) holds.

TABLE V

HEURISTIC PROTOCOL FOR INCREASING THE SOCIAL UTILITY OF RSG FOR CASE 2 OF MULTI-LEADER MULTI-FOLLOWER GAME

Start $n_L = 1$ and $n_L \in \mathcal{N}_L$

Consider $\mathcal{N}_{\rm F}^{\rm new} = \{1, \cdots, n_{L-1}, n_{L+1}, \cdots, N_L\} \bigcup \mathcal{N}_{\rm F}$,

For all players of Stackelberg game except leader n_L,

Calculate C7-C8 by considering $n_L = 0$ and $\mathcal{N}_{\rm F}^{\rm new}$ as the set of followers,

Calculate $\mathbf{C}_{n_{\rm L}} = \sum_{m \in \mathcal{N}_{\rm F}^{\rm new}} \mathbf{C}_{m n_{\rm L}}$,

If C7-C8 hold for n_L,

Play RSG with n_L as the leader and others as followers:

1. Leader n_L announces its strategy,
2. All players in $\mathcal{N}_{\rm F}^{\rm new}$ play the strategic game,
3. Break.

If $n_L = N_L + 1$,

Find the leader $n_{\rm L}$ such that $\mathbf{C}_{n_{\rm L}} > \mathbf{C}_m$ for all $m \in \mathcal{N}_{\rm L}$ and $m \neq n_{\rm L}$,

Play RSG with n_L as the leader and others as followers:

1. Leader n_L announces its strategy,
2. All players in $\mathcal{N}_{\rm F}^{\rm new}$ play the strategic game,
3. Break.

Otherwise set $n_L = n_L + 1$,

Continue

TABLE VI

PERFORMANCE OF THE HEURISTIC PROTOCOL

Utility	NE	NSE	RSE for Heuristic Algorithm
Leader 1	7.07	5.06	4.97
Leader 2	1.67	4.33	4.19
Follower	1.7	2.08	2.49
Social utility of Leader 2 and followers	3.37	6.41	6.68