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# User Subscription Dynamics in Communication Markets†

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***Abstract*—In order to understand the competition and interactions between different technologies, it is of fundamental importance to study how users select these technologies operated by different network service providers (NSPs) . In this paper, we study the dynamics of user subscription in a wireless communication market that includes a continuum of users. First, we focus on a monopoly market with only one NSP that provides to each user with an unreliable quality-of-service (QoS) subject to the number of subscribers. Users dynamically make their decisions regarding whether or not they subscribe to the NSP. It is shown that there exists a unique equilibrium point in the dynamics and that the dynamics is guaranteed to converge under some sufficient conditions that can be interpreted as that the provided QoS does not vary too rapidly with respect to the change of user subscriptions. Then, we extend the analysis to a duopoly market by adding into the market another NSP that has sufficient resources and thereby provides to each user a constant QoS.**

## I. INTRODUCTION

It is envisioned that in future communication markets, multiple heterogeneous wireless technologies (e.g., CDMA2000 and Wireless Regional Area Network using white spaces in the TV frequency spectrum), administered by different network service providers (NSPs), will be accessible to mobile devices that are capable of dynamically switching among these available technologies [1]. In order to understand the competition and interactions between different technologies, it is of fundamental importance to study how users select these technologies operated by different NSPs.

Recently, wireless communication market has been attracting an unprecedent amount of attention from various research communities, due to its rapid expansion. For example, [1] studied the market dynamics with the coexistence of next-generation networks and conventional networks, by applying a two-sided model that includes content providers, NSPs and users. Nevertheless, the model characterizing technologies was too simple to reflect some of the key characteristics, e.g., what level of quality of service (QoS) can a certain technology provide. By focusing on two specific access technologies (i.e., wide and local area network), the authors in [2] applied stochastic geometric model to capture the user distributions and studied the convergence of user subscription dynamics. The analysis, however, cannot be easily generalized to other settings to incorporate different technologies. Another paper on the dynamics of user subscription is [3] where the authors investigated the user evolution in wireless social community networks. A key assumption is that the social community network provides a higher QoS to each user as the number of subscribers increases. While this assumption may hold if the network coverage is the only factor that determines the QoS, it does not model the QoS degradation due to, for instance, user traffic congestions at the NSP. To reflect accurately the QoS, it is necessary to take into account other factors in the QoS model, e.g., throughput, delay and energy efficiency [2].

In what follows, we are interested in finding the impacts of communication technologies on the users' subscription choices and the NSP's market shares. First, we consider a monopoly wireless market with only one NSP that is only capable of providing to each user a subscription-dependent QoS, modeled as a general function. Given the provided QoS and charged price, users choose to subscribe to the NSP if and only if the NSP can provide a non-negative utility. By analyzing the impact of QoS functions on the user subscription dynamics, we show that, for any QoS function and price, the dynamic process of user subscription always admits a unique equilibrium, at which no user wishes to change its subscription decision. We further obtain a sufficient condition that the QoS function needs to fulfill to guarantee the convergence of the user subscription dynamics. The derived sufficient condition can be interpreted as that the QoS provided by the resource-constrained NSP should not degrade too fast when more users subscribe. Then, we extend the user subscription analysis to a duopoly market by adding into the market another NSP that has sufficient resources and thereby provides to each user a constant QoS.

The rest of this paper is organized as follows. Section II describes the system model. In Section III, we study the user subscription dynamics in both monopoly and duopoly markets. Finally, we conclude this paper in Section IV.

## II. SYSTEM MODEL

We consider a communication market in which two NSPs, represented by $\mathcal{S}_1$ and $\mathcal{S}_2$, operate. There are a continuum of users, as in [3], that can potentially subscribe to one of

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the NSPs for communication services. The continuum model approximates well the real user population if there are a sufficiently large number of users in the market so that each individual user has negligible effects. As in [2][3], we assume throughout this paper that each user can subscribe to at most one NSP at any time instant. We also assume that NSP $\mathcal{S}_1$ has sufficient resources to provide a guaranteed level of QoS to all of its subscribers [3], whereas NSP $\mathcal{S}_2$ is resource-constrained and thus is prone to congestion among subscribers.[1] In other words, the QoS provided by NSP $\mathcal{S}_1$ is the same regardless of the number of its subscribers, whereas the QoS provided by NSP $\mathcal{S}_2$ degrades with the number of its subscribers [2]. Let $\lambda_i$ be the fraction of users subscribing to $\mathcal{S}_i$ for $i = 1, 2$. Then $\lambda_1$ and $\lambda_2$ satisfy $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 \leq 1$. Also, let $q_i$ be the QoS provided by NSP $\mathcal{S}_i$ for $i = 1, 2$. Note that $q_1$ is independent of $\lambda_1$ while $q_2$ is non-increasing in $\lambda_2$. We use a function $g(\cdot)$ defined on $[0, 1]$ to express the QoS provided by $\mathcal{S}_2$ as $q_2 = g(\lambda_2)$.

Users are heterogeneous in the sense that they may value the same level of QoS differently. Each user $k$ is characterized by a scalar $\alpha_k$, which represents its valuation of QoS. Specifically, when user $k$ subscribes to NSP $\mathcal{S}_i$, its utility is given by

$$u_{k,i} = \alpha_k q_i - p_i, \tag{1}$$

where $p_i$ is the subscription prices of NSP $\mathcal{S}_i$, for $i = 1, 2$.[2] Users that do not subscribe to either of the two NSPs get a utility of zero. Note that in our model the NSPs are allowed to engage in neither QoS discrimination nor price discrimination. That is, all users subscribing to the same NSP receive the same QoS and pay the same subscription price [3].

Now, we impose assumptions on the QoS function of NSP $\mathcal{S}_2$, user subscription choices, and the valuation of QoS as follows.

*Assumption 1:* $g(\cdot)$ is a non-increasing and continuously differentiable[3] function and $0 < g(\lambda_2) < q_1$ for all $\lambda_2 \in [0, 1]$.

*Assumption 2:* Each user $k$ subscribes to $\mathcal{S}_i$ if $u_{k,i} > u_{k,j}$ and $u_{k,i} \geq 0$ for $i, j \in \{1, 2\}$ and $i \neq j$. If $u_{k,1} = u_{k,2} > 0$, user $k$ subscribes to NSP $\mathcal{S}_1$.[4]

*Assumption 3:* The users' valuation of QoS follows a probability distribution whose probability density function (PDF) $f(\cdot)$ is strictly positive and continuous on $[0, \beta]$ for some $\beta > 0$ [3]. For completeness of definition, we have $f(\alpha) = 0$ for all $\alpha \notin [0, \beta]$. The cumulative density function (CDF) is given by $F(\alpha) = \int_{-\infty}^{\alpha} f(x)dx$ for $\alpha \in \mathbb{R}$.

We briefly discuss the above three assumptions. Assumption 1 captures the congestion effects that users experience when subscribing to the resource-constrained NSP $\mathcal{S}_2$ (e.g., traffic congestion in [2]). The shape of the QoS function $g(\cdot)$ of NSP $\mathcal{S}_2$ is determined by various factors including the resource allocation scheme and the scheduling algorithm of $\mathcal{S}_2$. Assumption 2 can be interpreted as rational subscription decisions. A rational user will subscribe to the NSP that provides a higher utility if at least one NSP provides a non-negative utility, and to neither NSP otherwise. Assumption 3 can be considered as an expression of user diversity in term of the valuation of QoS. When the users' valuation of QoS is sufficiently diverse, its distribution can be described by a continuous positive PDF on a certain interval as in [3].

## III. User Subscription Dynamics

In this section, we study the user subscription dynamics under the model introduced in Section II. For the convenience of presentation, we shall first focus on the monopoly market (i.e., with only NSP $\mathcal{S}_1$ or $\mathcal{S}_2$) and then extend the analysis to the duopoly market.

### A. User Subscription in Monopoly Market

When only NSP $\mathcal{S}_2$ operates in the communication market, each user has a choice of whether to subscribe to NSP $\mathcal{S}_2$ or not at each time instant. Since the QoS provided NSP $\mathcal{S}_2$ is varying with the fraction of subscribers, each user will form a belief, or expectation, about the fraction of subscribers when it makes a subscription decision. To describe the dynamics of user subscription, we construct and analyze a dynamic model which specifies how users form their beliefs and make decisions based on their beliefs. We consider a discrete-time model with time periods indexed $t = 1, 2, \ldots$. At each period $t$, each user $k$ holds a belief on $\lambda_2$, denoted by $\tilde{\lambda}_{2,k}^t$, and make a subscription decision in a myopic way to maximize its expected utility in the current period.[5] Then user $k$ subscribes to NSP $\mathcal{S}_2$ at period $t$ if and only if $\alpha_k g(\tilde{\lambda}_{2,k}^t) \geq p_2$. We specify that every user expects that the fraction of subscribers in the current period is equal to that in the previous period. That is, we have $\tilde{\lambda}_{2,k}^t = \lambda_2^{t-1}$ for $t = 1, 2, \ldots$, where $\lambda_2^0$ is the initial fraction of subscribers and $\lambda_2^t$ is the fraction of subscribers in period $t$.[6]

Given the description of our model, the fraction of subscribers follows a sequence $\{\lambda_2^t\}_{t=0}^{\infty}$ in $[0, 1]$ generated by

$$\lambda_2^t = h_m(\lambda_2^{t-1}) \triangleq 1 - F\left(\frac{p_2}{g(\lambda_2^{t-1})}\right), \tag{2}$$

for $t = 1, 2, \ldots$, starting from a given initial point $\lambda_2^0 \in [0, 1]$. Note that the price $p_2$ of NSP $\mathcal{S}_2$ is held fixed over time. Given the user subscription dynamics (2), we are interested in whether the fraction of subscribers will stabilize in the long

[1] An example that fits into our assumptions on NSPs is a cognitive radio network in which NSP $\mathcal{S}_1$ is a licensed operator serving primary users while NSP $\mathcal{S}_2$ is a spectrum broker serving secondary users. Another example is a network in which NSP $\mathcal{S}_1$ serves each user using a dedicated channel while $\mathcal{S}_2$ has its users share its limited resources or capacity.

[2] A similar quasilinear utility model can be found in [3][7].

[3] Since $g(\cdot)$ is defined on $[0, 1]$, we use a one-sided limit to define the derivative of $g(\cdot)$ at 0 and 1, i.e., $dg(0)/d\lambda_2 = \lim_{\lambda_2 \to 0^+} [g(\lambda_2) - g(0)]/(\lambda_2 - 0)$ and $dg(1)/d\lambda_2 = \lim_{\lambda_2 \to 1^-} [g(\lambda_2) - g(1)]/(\lambda_2 - 1)$.

[4] Specifying an alternative tie-breaking rule (e.g., random selection between the two NSPs) in case of $u_{k,1} = u_{k,2} \geq 0$ will not affect the analysis of this paper, since the fraction of indifferent users is zero under Assumption 3.

[5] An example consistent with our subscription timing is a "Pay-As-You-Go" plan in which a subscribing user pays a fixed service charge for a unit of time (day, week, or month) and is free to quit its subscription at any time period, effective from the next time unit.

[6] This model of belief formation is called naive or static expectations in [9]. A similar dynamic model of belief formation and decision making has been extensively adopted in the existing literature, e.g., [2][3][6].

run and, if so, to what value. As a first step, we define an equilibrium point of the user subscription dynamics.

*Definition 1:* $\lambda_2^*$ is an *equilibrium* point of the user subscription dynamics in the monopoly market of NSP $\mathcal{S}_2$ if it satisfies

$$h_m(\lambda_2^*) = \lambda_2^*. \tag{3}$$

Definition 1 implies that once an equilibrium point is reached, the dynamics stays there from that point on. Thus, equilibrium points can be considered as candidates for long-run stability points. The following Proposition, whose proof is deferred to [8], establishes the existence and uniqueness of an equilibrium point.

**Proposition 1.** *For any non-negative price $p_2$, there exists a unique equilibrium point of the user subscription dynamics in the monopoly market of NSP $\mathcal{S}_2$.* ■

Although Proposition 1 guarantees the existence of a unique equilibrium point, it does not provide us with an explicit expression of the equilibrium point as a function of the monopoly price. In order to obtain a closed-form expression of the equilibrium point, we consider a class of simple QoS functions defined below.

*Definition 2:* The QoS function $g(\cdot)$ is *linearly-degrading* if $g(\lambda_2) = \bar{q}_2 - c\lambda_2$ for all $\lambda_2 \in [0, 1]$, for some $\bar{q}_2 > 0$ and $c \in [0, \bar{q}_2)$. In particular, a linearly-degrading QoS function with $c = 0$, i.e., $g(\lambda_2) = \bar{q}_2$ for all $\lambda_2 \in [0, 1]$, is referred to as a constant QoS function.

Linearly-degrading QoS functions model a variety of applications including flow control and capacity sharing in [7]. It can also be viewed as the first-order Taylor approximation of a complicated QoS function. With a linearly-degrading QoS function and uniformly distributed valuations of QoS [3][5][7], we can obtain a simple closed-form expression of the equilibrium point. Specifically, with $g(\lambda_2) = \bar{q}_2 - c\lambda_2$ for $\lambda_2 \in [0, 1]$ and $f(\alpha) = 1/\beta$ for $\alpha \in [0, \beta]$, the equilibrium point of the user subscription dynamics in the monopoly market of NSP $\mathcal{S}_2$ can be expressed as a function of $p_2$ as follows:

$$\lambda_2^*(p_2) = \begin{cases} \frac{\bar{q}_2 + c - \sqrt{(\bar{q}_2 - c)^2 + \frac{4cp_2}{\beta}}}{2c} & \text{for } p_2 \in [0, \beta\bar{q}_2], \\ 0 & \text{for } p_2 \in (\beta\bar{q}_2, \infty), \end{cases} \tag{4}$$

if $c \in (0, \bar{q}_2)$ and $\lambda_2^*(p_2) = \max\{0, 1 - p_2/(\beta\bar{q}_2)\}$ if $c = 0$.

Our equilibrium analysis so far guarantees the existence of a unique stable point of the user subscription dynamics in the monopoly market. However, it does not discuss whether the unique stable point will be eventually reached. To answer this question, we turn to the analysis of the convergence properties of the user subscription dynamics. The convergence of the user subscription dynamics is not always guaranteed, especially when the QoS provided by the monopolist degrades rapidly with respect to the fraction of subscribers. As a hypothetical example, suppose that only a small fraction of users subscribe to NSP $\mathcal{S}_2$ at period $t$ and obtains a high QoS. In our model of belief formation, users believe that the QoS will remain high at period $t + 1$, and thus a large fraction of users subscribe at period $t + 1$, which will result in a low QoS at period $t + 1$. This in turn will induce a small fraction of subscribers at period $t + 2$. When QoS is very sensitive to the fraction of subscribers, the user subscription dynamics may oscillate away from the equilibrium point and thus convergence may not be obtained. The following theorem provides a sufficient condition under which the user subscription dynamics always converges.

**Theorem 1.** *For any non-negative price $p_2$, the user subscription dynamics specified by (2) converges to the unique equilibrium point starting from any initial point $\lambda_2^0 \in [0, 1]$ if*

$$\max_{\lambda_2 \in [0,1]} \left[ -\frac{g'(\lambda_2)}{g(\lambda_2)} \right] < \frac{1}{K}, \tag{5}$$

*where $K = \max_{\alpha \in [0,\beta]} f(\alpha)\alpha$.*

*Proof:* See the appendix. □

By applying Theorem 1 to linearly-degrading QoS functions, we obtain the following result.

**Corollary 1.** *If the QoS function $g(\cdot)$ is linearly-degrading, i.e., $g(\lambda_2) = \bar{q}_2 - c\lambda_2$ for $\lambda_2 \in [0, 1]$, then the user subscription dynamics converges to the unique equilibrium point starting from any initial point $\lambda_2^0 \in [0, 1]$ if*

$$\frac{c}{\bar{q}_2} < \frac{1}{1 + K}. \tag{6}$$

The condition (5) in Theorem 1 is sufficient but not necessary for the convergence of the user subscription dynamics. In particular, we observe through numerical simulations that in some cases (e.g., $g(\lambda_2) = 1 - 0.9\lambda_2$ for $\lambda_2 \in [0, 1]$ and $f(\alpha) = 1$ for $\alpha \in [0, 1]$) the user subscription dynamics converges for a wide range of prices although the condition (5) is violated. Nevertheless, the sufficient condition provides us with the insight that if QoS degradation is too fast (i.e., $-Kg'(\lambda_2)$ is larger than $g(\lambda_2)$ for some $\lambda_2 \in [0, 1]$), the dynamics may oscillate.

Consider a scenario where only a fraction of users, denoted by $\epsilon \in (0, 1]$, can change their subscription decisions in each period while the users form their beliefs as before. Then the user subscription dynamics is generated by

$$\lambda_2^t = (1 - \epsilon)\lambda_2^{t-1} + \epsilon h_m(\lambda_2^{t-1}) \tag{7}$$

for $t = 1, 2, \ldots$, starting from an initial point $\lambda_2^0 \in [0, 1]$. Note that (7) is more general than (2) since (7) reduces to (2) when $\epsilon = 1$. Definition 1 still gives the definition of an equilibrium point of the user subscription dynamics (7), and thus the equilibrium (Proposition 1) and convergence analysis (Theorem 1) are still valid.

Now we consider a monopoly market of NSP $\mathcal{S}_1$. Since NSP $\mathcal{S}_1$ can be considered as having a constant QoS function $g(\lambda_1) = q_1$, we obtain the following result regarding the monopoly market of NSP $\mathcal{S}_1$.

**Proposition 2.** *In the monopoly market of NSP $\mathcal{S}_1$, for any non-negative price $p_1$, the user subscription dynamics always* *converges to the unique equilibrium point* $\lambda_1^* = 1 - F(p_1/q_1)$ *starting from any initial point* $\lambda_1^0 \in [0,1]$. ■

### B. User Subscription in Duopoly Market

With the two NSPs operating in the market, each user has three possible choices at each time instant: subscribe to NSP $\mathcal{S}_1$, subscribe to NSP $\mathcal{S}_2$, and subscribe to neither. As in the monopoly market, we consider a dynamic model in which the users update their beliefs and make subscription decisions at discrete time period $t = 1, 2, \ldots$. The users expect that the QoS provided by NSP $\mathcal{S}_2$ in the current period is equal to that in the previous period and make their subscription decisions to myopically maximize their expected utility in the current period [2][3]. We assume that the users can switch between $\mathcal{S}_1$ and $\mathcal{S}_2$ without incurring any costs. By Assumption 2, at period $t = 1, 2\cdots$, user $k$ subscribes to NSP $\mathcal{S}_1$ if and only if

$$\alpha_k q_1 - p_1 \geq \alpha_k g(\lambda_2^{t-1}) - p_2 \text{ and } \alpha_k q_1 - p_1 \geq 0, \tag{8}$$

to NSP $\mathcal{S}_2$ if and only if

$$\alpha_k g(\lambda_2^{t-1}) - p_2 > \alpha_k q_1 - p_1 \text{ and } \alpha_k g(\lambda_2^{t-1}) - p_2 \geq 0, \tag{9}$$

and to neither NSP if and only if

$$\alpha_k q_1 - p_1 < 0 \text{ and } \alpha_k g(\lambda_2^{t-1}) - p_2 < 0. \tag{10}$$

Given the prices $(p_1, p_2)$, the user subscription dynamics in the duopoly market is described by a sequence $\{(\lambda_1^t, \lambda_2^t)\}_{t=0}^{\infty}$ in $\Lambda = \{(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 \mid \lambda_1 + \lambda_2 \leq 1\}$ generated by

$$\lambda_1^t = h_{d,1}(\lambda_1^{t-1}, \lambda_2^{t-1}) \triangleq 1 - F\left(\frac{p_1 - p_2}{q_1 - g(\lambda_2^{t-1})}\right), \tag{11}$$

$$\lambda_2^t = h_{d,2}(\lambda_1^{t-1}, \lambda_2^{t-1}) \triangleq F\left(\frac{p_1 - p_2}{q_1 - g(\lambda_2^{t-1})}\right) - F\left(\frac{p_2}{g(\lambda_2^{t-1})}\right) \tag{12}$$

if $p_1/q_1 > p_2/g(\lambda_2^{t-1})$, and by

$$\lambda_1^t = h_{d,1}(\lambda_1^{t-1}, \lambda_2^{t-1}) \triangleq 1 - F\left(\frac{p_1}{q_1}\right), \tag{13}$$

$$\lambda_2^t = h_{d,2}(\lambda_1^{t-1}, \lambda_2^{t-1}) \triangleq 0. \tag{14}$$

if $p_1/q_1 \leq p_2/g(\lambda_2^{t-1})$, for $t = 1, 2, \ldots$, starting from a given initial point $(\lambda_1^0, \lambda_2^0) \in \Lambda$. Note that there are two regimes of the user subscription dynamics in the duopoly market, and which regime governs the dynamics depends on the relative size of the *prices per QoS*, $p_1/q_1$ and $p_2/g(\lambda_2^{t-1})$.

We give the definition of an equilibrium point, which is similar to Definition 1.

*Definition 3:* $(\lambda_1^*, \lambda_2^*)$ is an *equilibrium* point of the user subscription dynamics in the duopoly market if it satisfies

$$h_{d,1}(\lambda_1^*, \lambda_2^*) = \lambda_1^* \text{ and } h_{d,2}(\lambda_1^*, \lambda_2^*) = \lambda_2^*. \tag{15}$$

We establish the existence and uniqueness of an equilibrium point and provide equations characterizing it in Proposition 3, whose proof is deferred to [8].

**Proposition 3.** *For any non-negative price pair* $(p_1, p_2)$*, there exists a unique equilibrium point* $(\lambda_1^*, \lambda_2^*)$ *of the user subscription dynamics in the duopoly market. Moreover,* $(\lambda_1^*, \lambda_2^*)$ *satisfies*

$$\begin{cases} \lambda_1^* = 1 - F\left(\frac{p_1}{q_1}\right),\ \lambda_2^* = 0, & \text{if } \frac{p_1}{q_1} \leq \frac{p_2}{g(0)}, \\ \lambda_1^* = 1 - F(\theta_1^*),\ \lambda_2^* = F(\theta_1^*) - F(\theta_2^*), & \text{if } \frac{p_1}{q_1} > \frac{p_2}{g(0)}, \end{cases} \tag{16}$$

*where* $\theta_1^* = (p_1 - p_2)/(q_1 - g(\lambda_2^*))$ *and* $\theta_2^* = p_2/g(\lambda_2^*)$. ■

Proposition 3 indicates that, given any prices $(p_1, p_2)$, the market shares of the two NSPs are uniquely determined when the fraction of users subscribing to each NSP no longer changes. It also shows that the structure of the equilibrium point depends on the relative values of $p_1/q_1$ and $p_2/g(0)$. Specifically, if the price per QoS of $\mathcal{S}_1$ is always smaller than or equal to that of $\mathcal{S}_2$, i.e., $p_1/q_1 \leq p_2/g(0)$, then no users subscribe to $\mathcal{S}_2$ at the equilibrium point. On the other hand, if $\mathcal{S}_2$ offers a smaller price per QoS to its first subscriber than $\mathcal{S}_1$ does, i.e., $p_1/q_1 > p_2/g(0)$, then both $\mathcal{S}_1$ and $\mathcal{S}_2$ may attract a positive fraction of the total user population subscribing to them.

We now investigate whether the user subscription dynamics specified by (11)–(14) stabilizes as time passes. As in the case of monopoly, the user subscription dynamics is guaranteed to converge to the unique equilibrium in the duopoly market when the QoS degradation of $\mathcal{S}_2$ is not too fast. In the following theorem, we provide a sufficient condition for convergence and the proof details can be found in [8].

**Theorem 2.** *For any non-negative price pair* $(p_1, p_2)$*, the user subscription dynamics specified by (11)–(14) converges to the unique equilibrium point starting from any initial point* $(\lambda_1^0, \lambda_2^0) \in \Lambda$ *if*

$$\max_{\lambda_2 \in [0,1]} \left[-\frac{g'(\lambda_2)}{g(\lambda_2)} \cdot \frac{q_1}{q_1 - g(\lambda_2)}\right] < \frac{1}{K}, \tag{17}$$

*where* $K = \max_{\alpha \in [0,\beta]} f(\alpha)\alpha$. ■

Note that the condition (17) imposes a more stringent requirement on functions $f(\cdot)$ and $g(\cdot)$ than the condition (5) does, since $q_1/(q_1 - g(\lambda_2)) > 1$ for all $\lambda_2 \in [0,1]$. However, the condition (17) provides us with a similar insight that, if QoS degradation is severe, the user subscription dynamics may exhibit oscillation. (see Fig. 1 for a graphical illustration).

## IV. Conclusion

In this paper, we studied the dynamics of user subscription in both monopoly and duopoly markets with a sufficiently large number of users. First, we considered a monopoly market with only one service provider that provides to each user an unreliable QoS subject to the number of subscribers. It was shown that there exists a unique equilibrium in the user subscription dynamics and that the equilibrium can be iteratively achieved under a certain sufficient condition. Then, we added into the market another NSP which has sufficient resources and

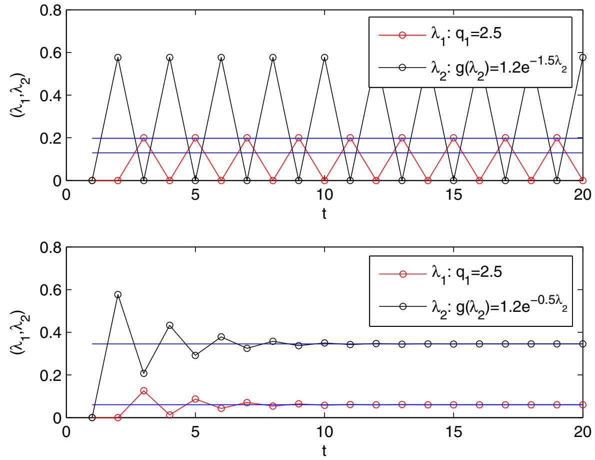


Fig. 1. Oscillation and convergence of user subscription dynamics. $p_1 = 2$, $p_2 = 0.6$, and $\alpha$ is uniformly distributed on $[0, 1]$, i.e., $f_a(\alpha) = 1$ for $\alpha \in [0, 1]$.

thereby provides to each user a constant QoS. Existence and uniqueness of the equilibrium in user subscription dynamics were proved. We also derived a sufficient condition for the convergence of the dynamics. The derived sufficient conditions for the user subscription convergence in both monopoly and duopoly markets provide us the insight that "in order for the user subscription dynamics to converge, the QoS of NSP $\mathcal{S}_2$ cannot be degrading too fast when more users subscribe." As part of our future work, we shall investigate the pricing strategies of NSPs that maximize their profits or revenues.

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## Appendix
## Proof of Theorem 1

We prove the convergence of the user subscription dynamics in the monopoly market based on the contraction mapping theorem.

*Definition 5 [10]:* A mapping $\mathbf{T} : \mathcal{X} \to \mathcal{X}$, where $\mathcal{X}$ is a closed subset of $\mathbb{R}^n$, is called a contraction if there is a real number $\kappa \in [0, 1)$ such that

$$\|\mathbf{T}(x_1) - \mathbf{T}(x_2)\| \le \kappa \|x_1 - x_2\|, \quad \forall\, x_1, x_2 \in \mathcal{X}, \tag{18}$$

where $\|\cdot\|$ is some norm defined on $\mathcal{X}$.

Proposition 1.1 in Chapter 3 of [10] shows an important property of a contraction mapping $\mathbf{T}$ that the update sequence generated by $x^{t+1} = \mathbf{T}(x^t)$, $t = 1, 2, \ldots$, converges to a fixed point $x^*$ satisfying $\mathbf{T}(x^*) = x^*$ starting from any initial value $x^0 \in \mathcal{X}$. To prove Theorem 1, we shall show that the function $h_m(\cdot)$, defined in (2), is a contraction mapping on $[0, 1]$ with respect to the absolute value norm if the condition (5) is satisfied.

Suppose $p_2 = 0$. Then $h_m(\lambda_2) = 1$ for all $\lambda_2 \in [0, 1]$, and thus $h_m(\cdot)$ is a contraction with $\kappa = 0$.

Suppose $p_2 > 0$. Let $\lambda_{2,a}$ and $\lambda_{2,b}$ be two different real numbers arbitrarily chosen from the interval $[0, 1]$, and suppose without loss of generality that $\lambda_{2,a} > \lambda_{2,b}$. We will show that

$$|h_m(\lambda_{2,a}) - h_m(\lambda_{2,b})| \le \kappa_m |\lambda_{2,a} - \lambda_{2,b}|, \tag{19}$$

where $\kappa_m = K \cdot \max_{\lambda_2 \in [0,1]}\{-g'(\lambda_2)/g(\lambda_2)\}$. Then the condition (5) implies that $\kappa_m \in [0, 1)$, establishing that $h_m(\cdot)$ is a contraction. Since $0 < p_2/g(\lambda_{2,b}) \le p_2/g(\lambda_{2,a})$, we can consider three cases.

*Case 1 ($p_2/g(\lambda_{2,a}), p_2/g(\lambda_{2,b}) < \beta$):* Note that $h_m$ is continuous on $[0, 1]$ and differentiable on $(\lambda_{2,b}, \lambda_{2,a})$. Hence, by the mean value theorem, there exists $\lambda_{2,c} \in (\lambda_{2,b}, \lambda_{2,a})$ such that

$$h'_m(\lambda_{2,c}) = \frac{h_m(\lambda_{2,a}) - h_m(\lambda_{2,b})}{\lambda_{2,a} - \lambda_{2,b}}. \tag{20}$$

Then we obtain

$$|h_m(\lambda_{2,a}) - h_m(\lambda_{2,b})| \tag{21}$$

$$= \left| f\left(\frac{p_2}{g(\lambda_{2,c})}\right) \frac{p_2}{g(\lambda_{2,c})} \frac{g'(\lambda_{2,c})}{g(\lambda_{2,c})} \right| |\lambda_{2,a} - \lambda_{2,b}| \tag{22}$$

$$\le \kappa_m |\lambda_{2,a} - \lambda_{2,b}|. \tag{23}$$

*Case 2 ($p_2/g(\lambda_{2,b}) < \beta \le p_2/g(\lambda_{2,a})$):* Let $\bar{\lambda}_2 = \min\{\lambda_2 \in [0, 1] |\ \beta g(\lambda_2) \le p_2\}$. Note that $\lambda_{2,b} < \bar{\lambda}_2 \le \lambda_{2,a}$. Applying the mean value theorem to $h_m(\cdot)$ on the interval $[\lambda_{2,b}, \bar{\lambda}_2]$ yields

$$|h_m(\bar{\lambda}_2) - h_m(\lambda_{2,b})| \le \kappa_m |\bar{\lambda}_2 - \lambda_{2,b}|. \tag{24}$$

Since $h_m(\bar{\lambda}_2) = h_m(\lambda_{2,a}) = 0$ and $\kappa_m \ge 0$, we obtain

$$|h_m(\lambda_{2,a}) - h_m(\lambda_{2,b})| \le \kappa_m |\bar{\lambda}_2 - \lambda_{2,b}| \le \kappa_m |\lambda_{2,a} - \lambda_{2,b}|. \tag{25}$$

*Case 3 ($p_2/g(\lambda_{2,a}), p_2/g(\lambda_{2,b}) \ge \beta$):* In this case, $h_m(\lambda_{2,a}) = h_m(\lambda_{2,b}) = 0$, and thus (19) is trivially satisfied. ■