THE APPENDIX FOR THE PAPER:
"INCENTIVE PROVISION AND JOB ALLOCATION IN SOCIAL CLOUD SYSTEMS"

Yu Zhang, Mihaela van der Schaar

Appendix A[1]

1) Proof of Proposition 1

Given the SCP π, each supplier's decision problem can be formulated as a continuous-time Markov decision process which is defined as follows:

- State $s \triangleq (\theta, k)$;
- Action a;
- Policy ω;
- State transition probability $f(s' \mid s, a) = p_{\omega}(\theta' \mid \theta, k) f_{\varphi}(k')$;
- Value function $U(\theta, k, \omega)$.

Since this Markov decision process has a finite state space and is ergodic, it is shown in [1] that there is always a unique optimal policy ω^*.

We then prove statement (i) and (ii). First, suppose that there is a pair (θ, k) such that $\omega_{\pi}^*(\theta, k) \in (0, k)$. According to (A1), we have that

$$U(\theta, k, \omega_{\pi}^*) = qk\psi(\theta) - c\omega_{\pi}^*(\theta, k) + \frac{\lambda_{\varphi}}{\lambda_{\varphi} + \delta} \sum_{\theta'} p_{\omega}(\theta' \mid \theta, k) V(\theta', \omega_{\pi}^*). \quad \text{(A1)}$$

If a supplier takes $a = 0$ instead of $\omega_{\pi}^*(\theta, k)$ when its state is (θ, k), $\{p_{\omega}(\theta' \mid \theta, k)\}$ does not change since the supplier will receive punishments whenever it deviates from $a = k$. Hence, its expected long-term utility becomes $qk\psi(\theta) + \frac{\lambda_{\varphi}}{\lambda_{\varphi} + \delta} \sum_{\theta'} p_{\omega}(\theta' \mid \theta, k) V(\theta', \omega_{\pi}^*)$ which is always higher than $U(\theta, k, \omega_{\pi}^*)$. According to the one-shot deviation principle, ω_{π}^* is not the optimal strategy, which leads to a contradiction. Therefore, it is proved that $\omega_{\pi}^*(\theta, k) \in \{0, k\}$ always holds.

Therefore to prove statement (ii), we only have to show that if $\omega_{\pi}^*(\theta, k) = 0$ for some θ, then $\omega_{\pi}^*(\theta', k) = 0$ for any $\theta' < \theta$.

[1] The equations in the appendix are numbered in the format (A#) in order to differentiate with the equations in the manuscript which are numbered in the format (#).

Suppose that $\omega_\pi^*(\theta,k)=0$ for some θ. Then according to the one-shot deviation principle, its expected long-term utility, i.e. $qk\psi(\theta)+\frac{\lambda_\varphi}{\lambda_\varphi+\delta}((1-\beta)V(\theta,\omega_\pi^*)+\beta V(0,\omega_\pi^*)$, is higher than the expected long-term utility when it chooses to play $a=k$, i.e. $qk\psi(\theta)-ck+\frac{\lambda_\varphi}{\lambda_\varphi+\delta}\sum_{\theta'}p(\theta'\mid\theta,k,k)V(\theta',\omega_\pi^*)$. Here $p(\theta'\mid\theta,k,a)$ denotes the reputation transition probability when the action a is chosen. Therefore, we have that $kc>\frac{\lambda_\varphi}{\delta+\lambda_\varphi}(1-\gamma_k)[\alpha V(\theta+1,\omega_\pi^*)+(\beta-\alpha)V(\theta,\omega_\pi^*)-\beta V(0,\omega_\pi^*)]$.

Without loss of generality, we assume that $\omega_\pi^*(\theta-1,k)=k$. Using the same argument, we have that $kc\le\frac{\lambda_\varphi}{\delta+\lambda_\varphi}(1-\gamma_k)[\alpha V(\theta,\omega_\pi^*)+(\beta-\alpha)V(\theta-1,\omega_\pi^*)-\beta V(0,\omega_\pi^*)]$. However, it has been proved in [2] that given the reputation update scheme (1) and the threshold-based pricing scheme ψ in the manuscript, it is always true that $V(\theta+1,\omega_\pi^*)\ge V(\theta,\omega_\pi^*),\ \forall\theta$ and also $V(\theta+1,\omega_\pi^*)-V(\theta,\omega_\pi^*)\ge V(\theta,\omega_\pi^*)-V(\theta-1,\omega_\pi^*),\ \forall\theta$. Therefore, we have

$$\alpha V(\theta+1,\omega_\pi^*)+(\beta-\alpha)V(\theta,\omega_\pi^*)-\beta V(0,\omega_\pi^*)\ge\alpha V(\theta,\omega_\pi^*)+(\beta-\alpha)V(\theta-1,\omega_\pi^*)-\beta V(0,\omega_\pi^*)\ .\tag{A2}$$

Hence, we have a contradiction and $\omega_\pi^*(\theta-1,k)=k$ cannot hold. Since this conclusion is valid for all θ, we should have $\omega_\pi^*(\theta',k)=0$ for any $\theta'<\theta$ if $\omega_\pi^*(\theta,k)=0$, and statement (ii) follows. ■

2) *Proof of Proposition 2*

This follows the same argument as Proposition 1. According to the one-shot deviation principle, an SCP is sustainable if and only if $U_{coop}(\theta,k)\ge U_{dev}(\theta,k)$ holds for any pair (θ,k), i.e.

$$qkI(\theta\ge h)-ck+\frac{\lambda_\varphi}{\delta+\lambda_\varphi}\sum_{\theta'}p_{coop}(\theta'\mid\theta,k)V(\theta,\omega_{coop})\ge qkI(\theta\ge h)+\frac{\lambda_\varphi}{\delta+\lambda_\varphi}((1-\beta)V(\theta,\omega_{coop})+\beta V(\max\{\theta-1,0\},\omega_{coop}))\ .\tag{A3}$$

With simple manipulations on (A3), we obtain (6) and Proposition 2 follows. ■

3) *Proof of Theorem 1*

Let

$$\Pi_{\theta,k}=\frac{\lambda_\varphi}{\delta+\lambda_\varphi}\frac{1-\gamma_k}{k}[\alpha V(\min\{\theta+1,L\},\omega_{coop})+(\beta-\alpha)V(\theta,\omega_{coop})-\beta V(0,\omega_{coop})]\tag{A4}$$

denote a supplier's incentive constraint at reputation θ, an SCP π is sustainable if and only if $\Pi_{\theta,k} \geq c$ for any θ and k.

First, it has been shown in [2] that given the reputation update scheme (1) and the threshold-based pricing scheme ψ in the manuscript, it is always true that $V(\theta+1,\omega_\pi^*) \geq V(\theta,\omega_\pi^*),\ \forall\theta$ and also $V(\theta+1,\omega_\pi^*) - V(\theta,\omega_\pi^*) \geq V(\theta,\omega_\pi^*) - V(\theta-1,\omega_\pi^*),\ \forall\theta$. Therefore, $\Pi_{\theta,k}$ monotonically increases with θ and by transforming the above condition (A4), we have that that an SCP π is sustainable only if $\Pi_{0,k} = \dfrac{\lambda_\varphi}{\delta+\lambda_\varphi}\dfrac{1-\gamma_k}{k}\alpha[V(1,\omega_{coop}) - V(0,\omega_{coop})] \geq c$ for any k, which is maximized when $h(\psi)=1$. Hence, it can be concluded that if an SCP π with $h>1$ is sustainable, then an SCP π' is also sustainable where its pricing threshold is $h=1$ and the other parameters are the same with π.

Now we look at the social welfare of a sustainable SCP. Since the social welfare is proportional to the fraction of suppliers whose reputations are no less than the threshold h, i.e. $\sum_{\theta\geq h}\eta_{\omega_{coop}}(\theta) = 1 - \sum_{\theta<h}\eta_{\omega_{coop}}(\theta)$. Hence, it is always true that $\sum_{\theta\geq h}\eta_{\omega_{coop}}(\theta) \leq 1 - \eta_{\omega_{coop}}(0)$. Therefore to summarize, if there is an SCP π with $h>1$ that is sustainable, we can always construct another sustainable SCP π' where its pricing threshold is $h=1$ and the other parameters are the same with π such that the social welfare under π' being $1-\eta_{\omega_{coop}}(0)$, which is higher than that under π. That is, the optimal sustainable SCP always has $h=1$.

Also, it should be noted that given a pricing threshold h, the expected one-stage game utility of a supplier whose reputation $\theta \geq h$ is always $(q-c)\int_{k_{\min}^\varphi}^{k_{\max}^\varphi} f_\varphi(k)k\,dk$, which is not influenced by the selection of L. Therefore, $\Pi_{0,k}$ remains unchanged when the value of L changes as long as $L \geq h$. We can thus conclude that if an SCP π' is also sustainable with its pricing threshold being $h=1$ and $L>1$, then an SCP π'' with $L=1$ and all the other parameters same as π' is also sustainable. Given the fact that the social welfare under π' and π'' are both $1-\eta_{\omega_{coop}}(0)$, we can conclude that if sustainable SCPs exist, then there is always an optimal SCP which delivers the highest social welfare with $h=1$ and $L>1$. Therefore, Theorem 1 follows. ■

Appendix B

1) Proof of Proposition 3

With two-level reputation, an SCP π is sustainable if and only if

$$kc \le \frac{\lambda_{\varphi}}{\delta+\lambda_{\varphi}}(1-\gamma_k)\beta[V(H,\omega_{coop})-V(B,\omega_{coop})], \tag{A5}$$

and

$$kc \le \frac{\lambda_{\varphi}}{\delta+\lambda_{\varphi}}(1-\gamma_k)\alpha[V(H,\omega_{coop})-V(B,\omega_{coop})]. \tag{A6}$$

Substituting (9) into (A5) and (A6), we have

$$\frac{kc}{1-\gamma_k} \le \frac{\lambda_{\varphi}}{\delta+\lambda_{\varphi}}\beta\frac{\overline{k}_{\varphi}q}{1-\frac{\lambda_{\varphi}}{\lambda_{\varphi}+\delta}[(1-\overline{\gamma}_{\varphi})(1-\alpha)+\overline{\gamma}_{\varphi}(1-\beta)]}, \tag{A7}$$

and

$$\frac{kc}{1-\gamma_k} \le \frac{\lambda_{\varphi}}{\delta+\lambda_{\varphi}}\alpha\frac{\overline{k}_{\varphi}q}{1-\frac{\lambda_{\varphi}}{\lambda_{\varphi}+\delta}[(1-\overline{\gamma}_{\varphi})(1-\alpha)+\overline{\gamma}_{\varphi}(1-\beta)]}. \tag{A8}$$

Hence, the minimum of the RHS of (A7) and (A8) is maximized when $\alpha=1$ and $\beta=1$ and consequently, when

$$\frac{kc}{1-\gamma_k} \le \frac{\lambda_{\varphi}}{\delta+\lambda_{\varphi}}\overline{k}_{\varphi}q. \tag{A9}$$

holds for all k, sustainable SCPs exist. Since (A9) equals to (10), Proposition 3 follows. ■

2) Proof of Theorem 2

From the manuscript, we have the average long-term utility of a supplier to be expressed as

$$V(H,\omega_{coop}) = \overline{k}_{\varphi}(q-c)+\frac{\lambda_{\varphi}}{\lambda_{\varphi}+\delta}(p_{coop}(H\mid H)V(H,\omega_{coop})+p_{coop}(B\mid H)V(B,\omega_{coop})) \tag{A11}$$

and

$$V(B,\omega_{coop}) = -\overline{k}_{\varphi}c+\frac{\lambda_{\varphi}}{\lambda_{\varphi}+\delta}(p_{coop}(H\mid B)V(H,\omega_{coop})+p_{coop}(B\mid B)V(B,\omega_{coop})). \tag{A12}$$

Here $p_{coop}(H\mid H)=\int_{k_{\min}^{\varphi}}^{k_{\max}^{\varphi}} p_{coop}(H\mid H,k)f_{\varphi}(k)dk = 1-\overline{\gamma}_{\varphi}+\overline{\gamma}_{\varphi}(1-\beta)$ and $p_{coop}(B\mid B)=\int_{k_{\min}^{\varphi}}^{k_{\max}^{\varphi}} p_{coop}(B\mid B,k)f_{\varphi}(k)dk = (1-\overline{\gamma}_{\varphi})(1-\alpha)+\overline{\gamma}_{\varphi}$. Meanwhile, we have

$\eta_{\omega_\pi^*}(H) = \frac{p_{coop}(H \mid B)}{p_{coop}(H \mid B) + p_{coop}(B \mid H)} = \frac{(1-\overline{\gamma}_\varphi)\alpha}{(1-\overline{\gamma}_\varphi)\alpha + \overline{\gamma}_\varphi \beta}$. Hence the optimal α_φ and β_φ can be solved in the following optimization problem

$$\begin{aligned}
&\max_{\alpha,\beta} \frac{(1-\overline{\gamma}_\varphi)\alpha}{(1-\overline{\gamma}_\varphi)\alpha + \overline{\gamma}_\varphi \beta} \\
&s.t. \quad \frac{kc}{1-\gamma_k} \le \frac{\lambda_\varphi}{\delta+\lambda_\varphi} \beta \frac{\overline{k}_\varphi q}{1 - \frac{\lambda_\varphi}{\lambda_\varphi+\delta}[(1-\overline{\gamma}_\varphi)(1-\alpha) + \overline{\gamma}_\varphi(1-\beta)]}, \forall k \\
&\qquad \frac{kc}{1-\gamma_k} \le \frac{\lambda_\varphi}{\delta+\lambda_\varphi} \alpha \frac{\overline{k}_\varphi q}{1 - \frac{\lambda_\varphi}{\lambda_\varphi+\delta}[(1-\overline{\gamma}_\varphi)(1-\alpha) + \overline{\gamma}_\varphi(1-\beta)]}, \forall k \;. \\
&\qquad 0 \le \alpha \le 1 \\
&\qquad 0 \le \beta \le 1
\end{aligned} \tag{A13}$$

Suppose $\beta_\varphi > \alpha_\varphi$, we then have

$$\frac{\lambda_\varphi}{\delta+\lambda_\varphi} \beta_\varphi \frac{\overline{k}_\varphi q}{1 - \frac{\lambda_\varphi}{\lambda_\varphi+\delta}[(1-\overline{\gamma}_\varphi)(1-\alpha_\varphi) + \overline{\gamma}_\varphi(1-\beta_\varphi)]} > \frac{\lambda_\varphi}{\delta+\lambda_\varphi} \alpha_\varphi \frac{\overline{k}_\varphi q}{1 - \frac{\lambda_\varphi}{\lambda_\varphi+\delta}[(1-\overline{\gamma}_\varphi)(1-\alpha_\varphi) + \overline{\gamma}_\varphi(1-\beta_\varphi)]} .$$

Hence, it is obvious that there always exist a sufficiently small Δ such that the SCP is still sustainable while the social welfare increases by replacing β_φ with $\beta_\varphi - \Delta$. This contradicts the fact that β_φ is the optimal SCP and hence we have $\beta_\varphi \le \alpha_\varphi$ always holds. It is always true that

$$\max_{k \in [k_{\min}^\varphi, k_{\max}^\varphi]} \frac{kc}{1-\gamma_k} = \frac{\lambda_\varphi}{\delta+\lambda_\varphi} \beta_\varphi \frac{\overline{k}_\varphi q}{1 - \frac{\lambda_\varphi}{\lambda_\varphi+\delta}[(1-\overline{\gamma}_\varphi)(1-\alpha_\varphi) + \overline{\gamma}_\varphi(1-\beta_\varphi)]}$$

and

$\frac{kc}{1-\gamma_k} \le \frac{\lambda_\varphi}{\delta+\lambda_\varphi} \alpha_\varphi \frac{\overline{k}_\varphi q}{1 - \frac{\lambda_\varphi}{\lambda_\varphi+\delta}[(1-\overline{\gamma}_\varphi)(1-\alpha_\varphi) + \overline{\gamma}_\varphi(1-\beta_\varphi)]}$. Using simple manipulation, it can be derived

that $\alpha_\varphi = 1$ and $\beta_\varphi = \max_{k \in [k_{\min}^\varphi, k_{\max}^\varphi]} \left\{ \frac{\frac{\delta+\lambda_\varphi}{\lambda_\varphi} \overline{k}_\varphi q - \gamma_k \overline{k}_\varphi q}{(1-\gamma_k)\overline{k}_\varphi q - \gamma_k kc} \right\}$. Hence, Theorem 2 follows. ■

3) Proof of Proposition 4

According to (10), it is straightforward that there are two regions $[\underline{k}_B, \overline{k}_B]$ and $[\underline{k}_H, \overline{k}_H]$ such that a supplier has sufficient incentive to cooperate if and only if the job load $k \in [\underline{k}_B, \overline{k}_B]$ when its reputation is B and $k \in [\underline{k}_H, \overline{k}_H]$ when its reputation is H. If $k < \underline{k}_B$, then the expected error probability $\gamma_k = 1-(1-\varepsilon)^{\overline{K}/k}$ is too large such that a supplier of reputation B loses its incentive to cooperate. On the other hand, if $k > \overline{k}_B$, the immediate cost for providing resources is too large such that a supplier of reputation B also do not want to cooperate. The same argument applies to $[\underline{k}_H, \overline{k}_H]$.

Hence, for a task of load K, it is optimal to divide the load into as few jobs as possible such that the failure probability of this task is minimized while the resulting load for each job falls in the regions $[\underline{k}_B, \overline{k}_B]$ and $[\underline{k}_H, \overline{k}_H]$. Hence, we have $K / \varphi^*(K) \le \min\{\overline{k}_B, \overline{k}_H\}$ and $K / (\varphi^*(K) - 1) > \min\{\overline{k}_B, \overline{k}_H\}$. Suppose $\varphi^*(K) > \varphi^*(K')$ with $K < K'$, it is obvious that $K' / \varphi^*(K') \ge K' / (\varphi^*(K) + 1) > K / (\varphi^*(K) + 1) > \min\{\overline{k}_B, \overline{k}_H\}$. Hence, suppliers will not cooperate in this task, which contradicts the claim that $\varphi^*(K')$ is the optimal job allocation for this task. ■

Appendix C

4) Proof of Theorem 3

Similar to Theorem 2, it can be shown that sustainable SCPHAs exist if and only if an SCPHA with $\alpha = 1$ and $\beta = 1$, which maximizes the minimum of the RHS of (A7) and (A8), is sustainable. Hence, the sufficient and necessary condition for the existence of sustainable SCPHAs is that the following inequality holds for some n

$$\frac{K_{max} c}{1-\gamma(n)} \le \frac{\lambda(n)}{\delta + \lambda(n)} \overline{K} q \,. \tag{A14}$$

By manipulating (A14), we have (14) and this theorem follows. ■

5) Proof of Theorem 4

This proposition can be proved in a similar manner as Theorem 2 and is omitted here. ■

6) Proof of Theorem 5

The upper bound of W^* is simply the optimal social welfare that can be achieved in the system where all suppliers fulfill their services and receive full payment from buyers. The lower bound is achieved by prescribing an SCPHA with $\alpha = 1$ and $\beta = 1$.

Consider a task of load $K \in [K_{min}, K_{max}]$. Since this SCPHA is sustainable, we should have the following inequality holds according to Proposition 3

$$(\frac{1}{(1-\varepsilon)^n} - 1)\frac{K}{n}c < \frac{\bar{K}}{n}q\,. \tag{A10}$$

Since this holds for all $K \in [K_{min}, K_{max}]$, we should have $n \le \ln(\frac{K_{max}c}{\bar{K}q + K_{max}c}) / \ln(1-\varepsilon)$. Also, given the fact that the social welfare under a sustainable SCPHA with $\alpha = 1$ and $\beta = 1$ is $(1-\gamma(n))\bar{K}q$, we have $\gamma(n) \le 1 - (1-\varepsilon)^{\ln(\frac{K_{max}c}{\bar{K}q+K_{max}c})/\ln(1-\varepsilon)}\bar{K}q$ and hence the lower bound follows. ■

REFERENCES

[1] A. Leizarowitz and A. J. Zaslavski, "Uniqueness and Stability of Optimal Policies of Finite State Markov Decision Processes", *Mathematics of Operations Research*, 32, pp. 156 – 167, 2007.

[2] Y. Zhang and M. van der Schaar, "Influencing the Long-term Evolution of Online Communities Using Social Norms," *Proc. Allerton Conference*, 2011.